DIFFERENTIAL AND
INTEGRAL CALCULUS

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WITHDRAWN

CORRECTIONS FOR COHEN'S DIFFERENTIAL AND INTEGRAL CALCULUS

IN THE TEXT

Page 6, line 15.	Replace “*than a previously*” by “*than any previously*”
p. 7, l. 4.	Replace “an” by “and”
p. 19, last l.	Replace $(y + y_0)$ by $(y - y_0)$
p. 21, ls. 11 and 10 fr. bott.	Replace “distance covered by” by “coördinate s of”
p. 21, l. 10 fr. bott.	Replace “in any time” by “at any time”
p. 36, formula III′.	Replace “$\boldsymbol{D^x u}$” by “$\boldsymbol{D_x u}$”
p. 45, l. 2 fr. bott.	Replace “$f(0) < f(\frac{1}{2})$” by “$f(0) > f(\frac{1}{2})$”
p. 47, l. 2.	Replace “2” by “$2\sqrt{2}$”
p. 48, l. 13.	Replace “$x^3 - 4a$” by “$x^3 - 4x$”
p. 49, l. 10.	Replace “$3x$” by “$6x$”
p. 50, l. 8.	Replace “$f'(y)$” by “$f'(x)$”
p. 51, l. 18.	Replace “$f(x)$” by “$f'(x)$”
p. 52, l. 7 fr. bott.	Replace “$f''(0)$” by “$f''(2)$”
p. 54, in Ex. 5.	Replace “8,” wherever it occurs, by “7”
p. 68, formula (7).	Replace “$x_o + x_o y_o'$” by “$x_o + y_o y_o'$”
p. 68, l. 2 fr. bott.	Replace “ordinate” by “abscissa”
p. 69, last l. in Ex. 6.	Replace “coördinate axes” by “point of contact”
p. 70, l. 14.	Replace “b” by “b^2”
p. 71, formula X.	Replace “$\phi(t)$” by “$\phi'(t)$”
p. 71, formula XI.	Replace “x^3” by “x'^3”

Page 72, line 7 fr. bott. Replace “$3ax$” by “$3axy$”

p. 79, l. 8 fr. bott. Replace “$v_x + v_y$” by “$v_x^2 + v_y^2$”

p. 82, l. 13. Replace “v” by “$|v|$”

p. 82, l. 16. Replace “$BO =$” by “$BQ =$”

p. 83, l. 3. Replace “$-\frac{v_o x}{r}$” by “$\frac{v_o x}{r}$”

p. 83, formula (9). Replace “$\frac{v_o}{r}$” by “$\frac{v_o^2}{r^2}$” in both places.

p. 87, in Ex. 2. Replace “$D_t A$” by “$D_r A$”

p. 101, in Ex. 1. Replace “subtangent” by “subnormal”

p. 104, l. 9. Replace “has to a constant area” by “has a constant area”

p. 107, l. 1. Replace “*integral of* x” by “*integral of* $f(x)$”

p. 123, Ex. 3. Replace “$x =$” by “$y =$”

p. 131, l. 20. Replace “larger” by “large”

p. 134, l. 3 fr. bott. Replace “$-\frac{2x}{x^2-1}$” by “$-\frac{2x}{x^2+1}$”

p. 144, l. 6. Replace “$\log \sec u$” by “$\log \sec x$”

p. 145, formula XVIII′. Replace “$d \tan^{-1}$” by “$d \tan^{-1} u$”

p. 147, l. 3. Replace “$\sin^{-1}$” by “$\sin^{-1} t$”

p. 148, l. 4. Replace “$\frac{a}{a-u}$” by “$\frac{a-u}{a}$”

p. 149, last l. Replace “$x + 1$” by “$x + 2$”

p. 153, formula (1). Replace the first “$\log (a - u) +$” by “$\log (a + u) -$”

p. 158, formula A. Replace “$\neq 1$” by “$\neq -1$”

p. 159, l. 4. Replace “$(\csc u - \cot u)$” by “$(\csc u + \cot u)$”

p. 161, l. 8 fr. bott. Replace “$\frac{a^2+b^2}{a^2}\int e^{ax} \cos bx$” by “$\frac{a^2+b^2}{b^2}\int e^{ax} \cos bx\, dx$”

P. 161, l. 7 fr. bott.	Replace "$\int e^{ax} \cos bx$" by "$\int e^{ax} \cos bx\, dx$"
p. 161, l. 8.	Replace "$\cos b$" by "$\cos bx$"
p. 162, l. 7.	Replace "$dv =$" by "$v =$"
p. 162, l. 10.	Replace "$\frac{2}{a}$" by "$\frac{1}{a}$"
p. 166, l. 6 fr. bott.	Replace "$(x + 1)(x - 3)$" by "$(x - 1)(x + 3)$." The resulting modifications are obvious.
p. 167, l. 10.	Replace "$A = -1$" by "$A = 1$"
p. 169, l. 4 fr. bott.	Replace "$-8x$" by "$-8x^2$"
p. 172, l. 11.	Replace "$+2\int$" by "$-2\int$"
p. 172, ls. 14, 15.	Replace "+ tan . . ." by "− tan . . ."
p. 177, l. 5 fr. bott.	Replace "$(x^1 - 1)^6$" by "$(x^{\frac{1}{6}} - 1)^6$"
p. 178, l. 6.	Replace "$(u^2 - a^2)^2$" by "$(u^2 - a)^2$"
p. 179, ls. 3, 4, 5.	Replace "$u^2 + 6x + 8$" by "$u^2 + 6u + 8$"
p. 179, l. 9.	Replace "$1 + u^2$" by "$1 + 4u^2$"
p. 179, l. 5 fr. bott.	Replace "$(ax + bu^{nk})$" by "$(a + bu^{nk})$"
p. 180, l. 2.	Add the factor "du" to the right hand member.
p. 185, l. 10.	Replace the exponent "$\frac{5}{3}$" by "$\frac{5}{2}$"
p. 189, l. 4.	Replace "$\dfrac{1 + \tan^3 \frac{x}{2}}{2 \tan \frac{x}{2}}$" by "$\dfrac{2 \tan \frac{x}{2}}{1 + \tan^3 \frac{x}{2}}$"
p. 192, l. 11.	Replace "$u = \tan x$" by "$u = \cot x$"
p. 194, l. 5 fr. bott.	Replace $\dfrac{\text{"}\sin (m - n)x\text{"}}{2(m + n)}$ by $\dfrac{\text{"}\sin (m - n)x\text{"}}{2(m - n)}$
p. 199, l. 3 fr. bott.	Add the factor "$\frac{\pi}{2}$" to the expression appearing there.

Page 204, lines 9, 11.	Replace “$u + \frac{\pi}{2}$” by “$u - \frac{\pi}{2}$”
p. 211, l. 15.	Replace “As a a rule” by “As a rule”
p. 214, l. 11 fr. bott.	Replace “1.2667, evidently too large.” by “1.1, a fairly good approximation.”
p. 220, l. 2 fr. bott.	Replace “$= \rho$” by “$= a\theta^2$”
p. 221, l. 5.	Replace “$= \theta$” by “$= a$”
p. 225, l. 1.	Replace “$= (a + b \sin \theta)$” by “$= (a + b) \sin \theta$”
p. 232, l. 10.	Replace “$s = \int$” by “$s = 2a\int$”
p. 239, l. 10.	Replace “$\frac{2y_o}{x_o}$” by “$\frac{2y_o}{x_o^2}$”
p. 243, l. 5 fr. bott.	Replace “$N(x)$” by “$N(y)$”
p. 247, formula (17).	Replace “$(\mu t + A)$” by “$(\mu t + \psi)$”
p. 259, formula (3).	Replace “$\frac{k^2}{gt}$” by “$\frac{gt}{k^2}$”
p. 264, l. 4. p. 265, l. 4. p. 266, l. 6 fr. bott.	In each case replace “e^c” by “$\log c$”
p. 284, l. 7.	Replace “$\sqrt{5}$” by “$\frac{1}{\sqrt{5}}$”
p. 285, l. 2 of Ex. 1.	Replace “$a \sin \theta$” by “$b \sin \theta$”
p. 287, ls. 7 and 6 fr. bott.	Replace “of these curves” by “, besides the origin,”
p. 295, l. 10.	Omit the limits from both integrals on this line.
p. 306, l. 2 of Ex. 6.	Replace “$8z^2$” by “$8z$”
p. 318, l. 10 fr. bott.	Replace “$\frac{2}{\pi a}$” by “$\frac{2}{\pi}$”
p. 330, formula (6).	Replace “$f'(z)$” by “$f'(v)$”
p. 330, Ex. 11.	Replace “$3xy$” by “$3xz$”

Page 334, line 13.	Replace “a $\phi''(x + ay) + a\,\psi''(x - ay)$” by “$a^2\phi''(x + ay) + a^2\psi(x - ay)$”
p. 335, Ex. 10.	Replace “$+ y^2u_y''$” by “$+ y^2u_{yy}''$”
p. 336, l. 12.	Replace “$2y^3$” by “$2y^3z$”
p. 343, l. 6.	Replace “3.7 %” by “2.9 %”
p. 343, l. 10.	Replace “average” by “relative”
p. 352, l. 10.	Replace second formula by “$\dfrac{\partial z}{\partial y} = -\dfrac{hx + by + fz}{gx + fy + cz}$”
p. 352, l. 10.	In the value of $\dfrac{\partial z}{\partial x}$ replace “by” by “hy”
p. 364, l. 11 fr. bott.	Replace “$- \Delta y^2$” by “$+ \Delta y^2$”
p. 370, l. 11.	Replace “$R(x, y, z)dy$” by “$R(x, y, z)dz$”
p. 372, formula (1).	Replace “$f(x, y, z, a)$” by “$f(x, y, a)$”
p. 375, last line.	Interchange “$\cos^3\alpha$” and “$\sin^3\alpha$”
p. 376, l. 1 of Rem. 2.	Replace “(a)” by “a”
p. 379, last line.	Replace “$(a^2 + b^2)^2$” by “$a^2 + b^2$”
p. 384, l. 4 fr. bott.	Prefix a “$-$” before $(a^2 - b^2)\sin^3\theta$.
p. 384, l. 3 fr. bott.	Prefix a “$-$” before $(a^2 - b^2)\frac{1}{3}\sin\theta$.
p. 385, Ex. 3.	Replace “$a \sinh u$” by “$b \sinh u$”
p. 385, l. 6 fr. bott.	Replace “$2a$” by “$- 2a$”
p. 388, Ex. 2.	Interchange parts (a) and (b).
p. 391, last l.	Replace “$\sqrt{a^2 - a^2}$” by “$\sqrt{a^2 - x^2}$”
p. 394, l. 8.	Replace “curve” by “area”
p. 403, Ex. 5.	Replace “$\cos^2\theta$” by “$\cos 2\theta$”
p. 406, formula $(3''')$.	Should read $(2''')$
p. 413, Ex. 3.	Interchange the words “from” and “by”
p. 422, l. 12.	Replace “a” by “a^2”
p. 423, l. 9 fr. bott.	Replace “$(a^2 - z^2)^2$” by “$(a - z^2)^2$”
p. 426, l. 5.	Replace “$\dfrac{a\sqrt{2}}{3}$” by “$\dfrac{a^3\sqrt{2}}{3}$”

Page 431, line 4.	Replace “b” by “B”
p. 433, l. 5.	Interchange “ρ” and “z”
p. 433, l. 6.	Replace “$z\rho^2 dz d\rho$” by “$\rho^2 dz d\rho$”
p. 434, l. 4.	Replace “$\frac{8a}{5}$” by “$\frac{8a}{5\pi}$”
p. 437, l. 11.	Replace “arc” by “arch”
p. 454, formula (1).	Replace “$f'(x)$” by “$f'(x_1)$
p. 455, formula (1).	Replace $\frac{“f'(x)”}{\phi'(x)}$ by $\frac{“f'(x_1)”}{\phi'(x_1)}$
p. 462, series (2).	Replace “$\frac{(-1)^{n+1}}{2}$” by “$\frac{(-1)^{n+1}}{2^n}$”
p. 465, l. 3.	Replace “R_n” by “$\|R_n\|$”
p. 467, Ex. 1.	Replace $\frac{“x^2”}{3!}$ by $\frac{“x^3”}{3!}$
p. 467, Ex. 3.	Replace “$(-1)^{n+1}$” by “$(-1)^n$”
p. 474, Ex. 18.	Omit the second part.
p. 480, l. 6.	Replace “$1<x<1$” by “$-1<x<1$”
p. 495, l. 9 fr. bott.	Insert “$\frac{1}{2}xy -$” before the integral.
p. 516, last l.	Replace “$x_0<a$” by “$x_0 \leq a$”
p. 532, l. 12.	Replace “$4ax$” by “ax”
p. 541, formula 9.	Replace second “dx” by “du”
p. 546, formula 55.	Add closing bracket after “dx”
p. 549, formula 92.	Replace $\frac{“2(n+1)k”}{2n+1}$ by $\frac{“2n+1”}{2(n+1)k}$
p. 549, formula 93.	Replace “$\int \frac{dx}{X}$” by “$\int \frac{dx}{\sqrt{X}}$”
p. 552, formula 118.	Replace “$\sqrt{\frac{x}{a+b}}$” by “x”
p. 555, formula 151.	Enclose “$\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}$ in brackets.

IN THE ANSWERS

Page 9.	Answer to Ex. 14, § 42 should read $\dfrac{-a\sin\theta}{(1-\cos\theta)^2}$
p. 16.	Answer to Ex. 2, § 59 should read $\dfrac{\sin x}{384}\Big[-48\cos^7 x + 8\cos^5 x + 10\cos^3 x + 15\cos x\Big] + \dfrac{15x}{384}.$
p. 17.	Both answers in Ex. 28, § 61 should read $\log\,(e^x - e^{-x})$.
p. 21.	Answer to Ex. 7, § 93 should read $a^2(e^{x_1/a} - e^{-x_1/a})$.
p. 23.	Answer to Ex. 30, § 102 should read $\frac{1}{3}a^2\sqrt{2}$
p. 28.	Answer to Ex. 3, § 128 should read $\frac{64}{9}a^3$.
p. 28.	Answers to Ex. 5 (a), § 132 should read $\frac{1}{12}k\pi a^2b^2, \frac{1}{4}ka$.
p. 29.	Answer to Ex. 11, § 135 should read $\left(\dfrac{5a}{2(2\pi-1)}, \dfrac{5a}{2(2\pi-1)}\right)$.
p. 29.	Answer to Ex. 12, § 135 should read $\left(1, \frac{2}{5}\right)$.

Parts of the Text of

COHEN'S DIFFERENTIAL AND INTEGRAL CALCULUS

THAT MAY BE OMITTED IN A FIRST COURSE

*Items marked by * are more or less optional, depending upon the wishes of the instructor and upon the time at his disposal.*

*Items marked by ** may be taken up, in part or as a whole, as the time permits.*

The remaining items should, as a rule, be omitted.

Page 40.
* Remarks 1 and 2, pages 56–57.
* Discussion of the folium of Descartes, pages 71–72.
Part of § 26, following equation (4), page 81 to equation (10), page 83.
§ 31, excepting the first nine lines.
* § 47.
Case 4, § 53.
** Types *B* and *C* of § 54. Exercises on page 182, and Exs. 41, 48 of § 61.
* § 57. Ex. 101, § 61.
* § 60. * § 64. * § 69. § 70.
Pages 238–240, and Ex. 1, page 241.
** Chapter IX.
Page 334. Exs. 7–12, page 335.
§ 108.
§§ 110–114.
Chapter XIII.
§ 126. § 147. * § 149. §§ 150–154.
** Appendix I, II, III.

DIFFERENTIAL AND INTEGRAL
CALCULUS

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D. C. HEATH AND COMPANY
BOSTON NEW YORK CHICAGO LONDON
ATLANTA DALLAS SAN FRANCISCO

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PRINTED IN U.S.A.

PREFACE

In the preparation of this book, the author has had in mind the needs of both colleges and technical schools. An attempt is made to give the student a working knowledge of the subject. Much emphasis is laid upon applications to geometry, mechanics, and the physical sciences. Without venturing too far into technical fields, of which the subject matter and terms might be unfamiliar to the student, the fact that the calculus has much to do with the laws of actual phenomena is abundantly illustrated. On the other hand, thorough training in the formal processes of the calculus is essential, and exercises adequate for this purpose have been included.

The book contains more material than is necessary for the usual first course in the subject. It has been the observation of the author that it is much more satisfactory to the student to omit, if necessary, certain parts of a comprehensive text, rather than have supplied to him from outside sources any missing material. It is believed that the book here presented contains the topics needed for building a satisfactory elementary course, either for students in pure mathematics, or for those intending to use the calculus in the sciences.

Rigorous proofs beyond the comprehension of the average student are not attempted. But it is hoped that the proper accent has been placed in the presentation of the various important fundamental ideas.

The division of the subject into differential and integral calculus has been disregarded.

The author takes this occasion to express his obligation to Professor Ira M. DeLong, of the University of Colorado, and

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to Mr. G. Y. Rainich, Johnston Scholar of the Johns Hopkins University, who have read the proofs and have made valuable suggestions; and to Lee Bren Cohen, who aided him materially in the preparation of the manuscript for the press. He desires also to express his appreciation of the coöperation of the publishers.

Abraham Cohen

The Johns Hopkins University,
Baltimore, Maryland,
May, 1925.

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DIFFERENTIAL AND INTEGRAL CALCULUS

CHAPTER I

FUNDAMENTAL PRINCIPLES

1. Variable; function. If A is the area of a circle and r its radius, we know that

$$A = \pi r^2, \tag{1}$$

where π is a fixed number or *constant*, whose value is approximately 3.1416. This formula gives the area of a circle whenever the radius is known.

Here the quantity r may have any positive value. It is said to be a *variable.* For a quantity is said to be a *variable* if its value is not fixed but may, in the course of a discussion, be any one of a set of numbers.

Since the value of A changes with that of r, A also is a variable; and besides, since its value depends upon that of r, it is said to be the *dependent* variable in the equation (1), while r is said to be the *independent* variable or *argument.* Instead of "dependent variable" the term "function" also is used very frequently. But, in the Calculus, the term "function" is used in a more general sense, as may be seen from the following definition.

A *function* of a variable is a quantity whose value is determined[1] whenever a definite value is assigned to the variable. Thus A is a function of r in (1).

---

[1] That this definition is more general than saying that "a function depends upon the value of the variable for its value" is obvious on noting

Again if v is the volume of a perfect gas, in a container, subject to a pressure p, then for varying pressure,

$$pv = k. \tag{2}$$

Writing this in the form

$$v = \frac{k}{p}, \tag{3}$$

k is a constant (so long as the temperature is constant; it is really a function of the temperature),

p is the variable[1] (or argument),

v is the function (or dependent variable).

If we write (2) in the form

$$p = \frac{k}{v}, \tag{3}$$

v is the variable and p the function.

This reciprocal relation between variable and function is not unusual. From the nature of the problem, however, the choice as to which is to be taken as variable is frequently determined.

EXERCISES

1. Express the area of a square as a function of a side.

2. Express the area of an equilateral triangle as a function of a side.

3. Two adjacent sides of a parallelogram are 3 and 5 inches respectively. Express the area of the parallelogram as a function of the angle between those sides.

4. Express the surface of a sphere as a function of the radius; also express the radius as a function of the surface.

---

that by it a constant, such as 2, is defined as a function of any variable, since its value is known corresponding to any value assigned to the variable; but its value does not depend upon any particular value assigned to the variable.

[1] In contradistinction to "function," "variable" means "independent variable." As a rule, when there is no likelihood of confusion, we shall refer to the independent variable as "the variable."

5. Express the volume of a sphere as a function of the surface.

6. Express the surface and volume of a cube as functions of an edge.

7. Express the surface and volume of a regular tetrahedron as functions of an edge.

2. Functional notation. Occasionally the student is confused by the symbolism employed when dealing with general functions. As a matter of fact this is a very simple matter:

We shall frequently let $f(x)$ [read "f of x"] represent any type of function of x we may have occasion to consider, when we do not wish to call attention to a particular function. The letter f is used because it is the initial one in the word *function.*

Equally well we could and do use $F(x)$, $\phi(x)$ [ϕ being the Greek letter corresponding to f], $g(x)$ [g being a neighboring letter to f], and so on. These several designations are especially serviceable when we have occasion to consider several distinct functions at the same time. In any discussion involving a single function, that function may be represented by $f(x)$, or $F(x)$, or $\phi(x)$, or any other convenient symbol. The fact to be noted is that *the behavior of a function depends upon its definition and not upon the symbol representing it.*

If $f(x)$ represents a function, $f(x_0)$ is used to represent the particular value which $f(x)$ has when $x = x_0$. Thus if $f(x) = x^2$, then $f(1) = 1^2 = 1$; $f(3) = 3^2 = 9$; $f(-2) = (-2)^2 = 4$; $f(a+b) = (a+b)^2 = (a^2 + 2ab + b^2)$; $f(\sqrt{2}) = (\sqrt{2})^2 = 2$; $f(\sin a) = \sin^2 a$; and so on. If on the other hand we had used ϕ instead of f, so that $\phi(x) = x^2$, then $\phi(1) = 1$; $\phi(3) = 9$; and so on.

If $F(t) = t^2 - 3t + 2$, then $F(0) = 2$; $F(1) = 0$; $F(a+b) = a^2 + 2ab + b^2 - 3a - 3b + 2$; and so on.

EXERCISES

1. If $f(x)=\frac{2}{x^2}$, evaluate $f(1)$, $f(2)$, $f(-1)$, $f(\frac{1}{2})$, $f(\sqrt{2})$.

2. If $f(a)=\sin a$, evaluate $f(0)$, $f\left(\frac{\pi}{3}\right)$, $f\left(\frac{\pi}{2}\right)$, $f\left(\frac{\pi}{2}-x\right)$.

3. If $\phi(t)=16t^2$, evaluate $\phi(0)$, $\phi(1)$, $\phi(10)$, $\phi(\frac{3}{4})$.

4. If $F(r)=\frac{r}{\sqrt{r^2+1}}$, evaluate $F(0)$, $F(1)$, $F(-2)$, $F(\frac{3}{4})$, $F(\tan a)$.

5. If $F(x)=\tan x$, evaluate $F(0)$, $F\left(\frac{\pi}{4}\right)$, $\frac{F(x)+F(y)}{1-F(x)F(y)}$.

3. Graph of a function. By the graph of a function $f(x)$ is meant the curve $y=f(x)$, where x and y are rectangular coördinates. Thus if $f(x)=ax^2$, where a is given a number, its graph is the parabola $y=ax^2$, Fig. 1.

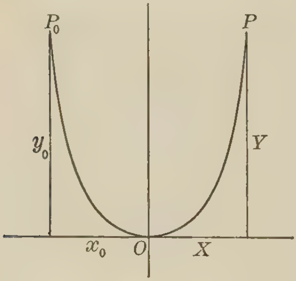

FIG. 1

The purpose of the graph is to illustrate the behavior of the function. From the figure we can tell readily the value of y (*i.e.* of the function) corresponding to any given value of x the variable.

But the graph tells us more: If we start with P_0, the point of the graph farthest to the left (*i.e.* the one determined by x_0, the algebraically smallest value of the variable for which the graph is given in Fig. 1), and follow the graph continuously toward the right, that is, let the variable grow steadily from x_0 to X, the largest value of x for which the graph is plotted, we see that y first diminishes until it reaches its smallest value at O, and after that increases.

Again, the graph of the function $\frac{4}{x}$ is the equilateral hyperbola $y=\frac{4}{x}$, Fig. 2. Starting with P_0, the point of the graph farthest to the left, and following the graph continuously, y diminishes until we reach P_1, the last point plotted for negative values of x. It is evident that no matter how large the space allotted to the figure or how small the unit selected, the graph will always extend beyond the allotted space as x gets nearer and nearer to zero. In the same way, no matter what be the space allotted, or the unit, no part of the graph will appear corresponding to very small positive values of x. Suppose that P_2 is the first point of the graph given in Fig. 2 as x grows from zero, and that P is the last one in the figure. Here also y is diminishing all the time.

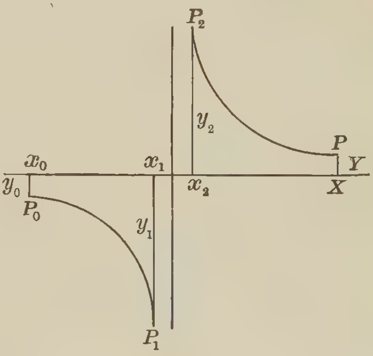

FIG. 2

The graph serves admirably to illustrate this "qualitative" property of the function referring to whether the function increases or decreases as the variable increases.

It also tells us in a general way in the neighborhood of which values of the variable the function increases or decreases most rapidly, corresponding to a given change in the variable. Thus in Fig. 1, x^2 grows most slowly in the neighborhood of $x = 0$, and changes more and more rapidly as x increases in absolute value. In Fig. 2, the opposite is true: $\frac{4}{x}$ diminishes more slowly the larger x is in absolute value, and changes more rapidly the nearer x is to zero.

But if we ask for the "quantitative" property of the change in the function, viz. the rate of change of the function with respect to that of the variable, we raise an arithmetical or analytic problem, which also is illustrated by the graph, as we shall see later (§ 24), and is one of the fundamental problems of the Calculus.

4. Limit. The notion of a limit is fundamental in the Calculus. In general the independent variable is unrestricted and not subject to any law of variation. When we say $y = x^2$ represents a parabola and try to find a sufficient number of points on it to sketch the curve more or less accurately, there need be no particular order in which we assign values to x and then find corresponding values of y. But when we say that *the independent variable is to approach a limit a*, while we still allow a wide range of choice of values for it, we restrict these choices in this way: Some law of choice of values must be established, such that after a namable stage in it, *all the values of the variable selected must differ from the limit a by less than a previously assigned positive amount ϵ.*

Such a choice of successive values of x may be the following: Starting with x_1, any convenient number either larger or smaller than a, take for the second selected value $x_2 = \frac{a + x_1}{2}$; then for the third, $x_3 = \frac{a + x_2}{2}$, which equals $\frac{3a + x_1}{4}$; for the next, $x_4 = \frac{a + x_3}{2}$, which in turn equals $\frac{7a + x_1}{8}$; and so on.

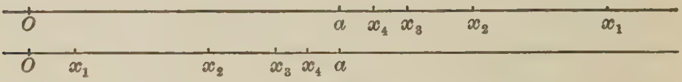

FIG. 3

This choice of values of the variable is illustrated in Fig. 3, where x_2 is the point midway between x_1 and a; x_3 is midway between x_2 and a; x_4 is midway between x_3 and a; and so on.

For the nth choice we have $x_n = \frac{a + x_{n-1}}{2}$, which equals

$\frac{(2^{n-1}-1)a+x_1}{2^{n-1}}$. That is, the difference between a and x_n is equal to the difference between a and x_1 divided by 2^{n-1}. Since the latter difference is fixed in value and 2^{n-1} grows with n, an n can be named so large that the difference between a and x_n will be smaller than any small positive number ϵ that has been assigned. From this stage on, that is, for all larger values of n, this difference remains less than the assigned ϵ.

A convenient way to represent the absolute value of a quantity h is by $|h|$; thus $|3|=3$, $|-3|=3$, $|1-2|=1$. With this notation, we say the above choice of successive values of x causes it to *approach a as a limit* because, corresponding to any given positive number ϵ, no matter how small, $|a-x_n|<\epsilon$ "for n sufficiently large and *all* larger values." In order to avoid this more or less involved phrase we shall say simply that *ultimately* $|a-x_n|<\epsilon$.

In case $x_1<a$, x is said to approach its limit "from the left." If $x_1>a$, x approaches its limit "from the right."

Other laws of approach can be suggested, among them instances in which x is sometimes greater and sometimes less than its limit, as in the case of the bob of a simple pendulum swinging in a resisting medium, where the limiting position is that in which the pendulum is at rest.

But we are not concerned with the way in which x approaches its limit. The essential fact connected with x approaching the limit a is that ultimately x will always differ from a by less than any desired amount. This may be stated concisely, in symbolic form, thus: If the successively selected values of x are $x_1, x_2, x_3, \cdots, x_n, \cdots$, then *ultimately we must have* $|a-x_n|<\epsilon$, *where ϵ is any assigned small positive quantity.*

Passing now to the consideration of the function $f(x)$, if, as x approaches its limit a, the successive values which $f(x)$ assumes *ultimately all* differ from some constant A by less than any assigned positive quantity ϵ, no matter how small, $f(x)$ is said to *have the limit* A (or to *approach* A *as a limit*) *as* x *approaches* a, and we shall designate this by

$$\underset{x \to a}{L} f(x) = A.$$

A convenient way of picturing this is the following: In order that $f(x)$ shall have the limit A as x approaches a, it must be possible to assure oneself that $f(x)$ will differ from A by as small an amount as one pleases so long as x differs from a by a sufficiently small amount.

In the Calculus we shall frequently be dealing with variables, both independent and dependent, whose law of variation is such that their limit is zero. Such a variable is said to be an *infinitesimal*. Thus, x is an infinitesimal, if we assign to it the successive values $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$. Corresponding to these values of x, $f(x) = \sin x$ is also an infinitesimal, because as x becomes smaller and smaller, approaching zero as a limit, so also does $\sin x$.

Using the language of infinitesimals, we may say that $f(x)$ *approaches the limit* A *as* x *approaches* a, *if* $f(x) - A$ *is an infinitesimal whenever* $x - a$ *is infinitesimal.* Thus, if we let x take successively the values $\frac{\pi}{2} - \pi, \frac{\pi}{2} - \frac{\pi}{2}, \frac{\pi}{2} - \frac{\pi}{3}, \cdots, \frac{\pi}{2} - \frac{\pi}{n}, \cdots$, it approaches $\frac{\pi}{2}$, and at the same time $\sin x$ approaches 1. Here $\sin x - 1$ is an infinitesimal along with $x - \frac{\pi}{2}$.

REMARK.— Attention should be called to the fact that an infinitesimal is, in general, a variable and need not always be a small quantity. In the early stages it may be quite large. The only requirement is that ultimately it must become and remain less in absolute value than any assigned positive number. A small constant (even though negligible for certain purposes of approximation) is not an infinitesimal in the mathematical sense. The *only* constant infinitesimal is the number zero, for that evidently satisfies the requirement of having zero for a limit, just as any constant (looked upon as a function of any variable, see first footnote § 1) may be said to have itself as a limit.

If the choice of successive values of a variable is such that the values ultimately exceed *any* assigned positive number, no matter how large, the variable is said to *approach* ∞ or to *have* ∞ *as a limit.* Similarly if the values of the variable ultimately are less algebraically than *any* assigned negative number, no matter how small, the variable is said to *approach* $-\infty$.

While the language of limits is used here, there is no question of approaching the limit by ultimately differing from the latter by less than any desired amount, as in the case when a finite limit is approached. It simply means that the variable grows beyond all limit.

In the same way if, as x approaches a limit a (including $\pm\infty$), $f(x)$ increases beyond all limit in absolute value, we shall still use the form

$$\underset{x \to a}{L} f(x) = \pm\infty,$$

although there is no actual approach to a limit in the sense that $f(x) - (\pm\infty)$ is an infinitesimal, as is the case when $f(x)$ approaches a finite limit. Thus if x is given the successive values $1, 2, 3, \cdots, n, \cdots$, it is said to approach ∞. Corresponding to this

$$\underset{x \to \infty}{L} \sin\frac{1}{x} = 0 \quad\text{and}\quad \underset{x \to \infty}{L} x^2 = \infty.$$

Again,

$$\underset{x \to \pi/2}{L} \tan x = \pm\infty,$$

according as x approaches its limit from "the left" or "the right."

5. Continuous function. In the definition of $\underset{x \to a}{L} f(x) = A$, nothing is predicated as to what happens when $x = a$. Two cases are of special interest to us here.

1. $\underset{x \to a}{L} f(x) = f(a)$. This is the usual case with most of the functions we shall have occasion to use. In this case, $f(x)$ is said to be *continuous at a*.

A little consideration will show that *if $f(x)$ is continuous at every value of the variable in a given interval, the graph of the function* (whenever it can be drawn) *is unbroken.*

2. $\underset{x \to a}{L} f(x) = A$, while $f(a)$ has no meaning. Thus consider the limit of $f(x) = \frac{x^2 - 1}{x - 1}$, as x approaches 1. Here $f(1) = \frac{0}{0}$, which has no definite value. If we put $x = 1 + h$, then, provided $h \neq 0$,

$$f(1 + h) = \frac{1 + 2h + h^2 - 1}{1 + h - 1} = 2 + h.$$

By making h smaller and smaller in absolute value, this can be made to differ from 2 by less than any previously assigned quantity. Hence $\underset{x \to 1}{L} \frac{x^2 - 1}{x - 1} = 2$.

Functions of this sort, which are fractional in form, in which numerator and denominator are infinitesimals, arise frequently in the Differential Calculus. The evaluation of the limit of such a function may frequently be effected by getting rid of a common vanishing factor in numerator and denominator. Thus in the case of $f(x) = \frac{x^2 - 1}{x - 1}$, there is the common factor $x - 1$. For all values of x, *other than* $x = 1$, we may get rid of this common factor and write $f(x) = x + 1$, which evidently becomes 2 when $x = 1$. While this tells us that $\underset{x \to 1}{L} f(x) = 2$, it gives us no information in regard to $f(1)$, which is absolutely indeterminate.

Whenever the case 1, $\underset{x \to a}{L} f(x) = f(a)$, fails to hold, the function $f(x)$ is said to be *discontinuous at a*, or the function is said to have a *discontinuity at a*. Various types of discontinuity arise in the study of higher branches of Mathematics. The only types we are apt to meet in this course are the following:

(*a*) Since ∞ is no definite number or value, $f(x)$ is discontinuous at a if $\underset{x \to a}{L} f(x) = \pm \infty$. In this case the graph is evidently broken; compare Fig. 2 in § 3.

(*b*) In a case like 2 above, when $f(a)$ is not defined, but when $\underset{x \to a}{L} f(x)$ is the same when x approaches a "from the right" and "from the left," $f(x)$ also is said to be discontinuous at a; but the discontinuity is said to be *removable* by adding to the original definition of $f(x)$ the requirement that $f(a) = \underset{x \to a}{L} f(x)$.

Thus in the special case cited above, where $f(x) = \frac{x^2-1}{x-1}$, this definition fails to determine $f(1)$. If we define in addition $f(1) = \underset{x \to 1}{L} \frac{x^2-1}{x-1} = 2$, then $x = 1$ is no longer a discontinuity of the function.

(*c*) Still another type of discontinuity may be mentioned here. Consider, for example, the function $f(x) = \tan^{-1}\frac{1}{x}$. As we shall see later (§ 43), in order that a single value may be assigned to $\tan^{-1}z$ corresponding to a given value of z, it is customary to restrict $\tan^{-1}z$ to its range of "principal" values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ as z varies from $-\infty$ to $+\infty$. Besides, it should be noted that $\tan^{-1}\infty$ has no meaning, that is, $\tan^{-1}z$ is not defined for $z = \infty$; but its limits as z approaches $+\infty$ and $-\infty$ are perfectly definite; thus $\underset{z \to \infty}{L} \tan^{-1}z = \frac{\pi}{2}$ and $\underset{z \to -\infty}{L} \tan^{-1}z = -\frac{\pi}{2}$. Coming back now to our function $f(x) = \tan^{-1}\frac{1}{x}$, we note that $f(0)$ is undetermined; but as x approaches 0 "from the right" (§ 4) $\tan^{-1}\frac{1}{x}$ approaches $\frac{\pi}{2}$, while as x approaches 0 "from the left" $\tan^{-1}\frac{1}{x}$ approaches $-\frac{\pi}{2}$. The graph of $\tan^{-1}\frac{1}{x}$ is given in Fig. 4.

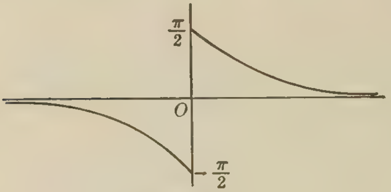

Fig. 4

The type of discontinuity illustrated here is where the function approaches definite but different values as the variable approaches a given limiting value a from different sides, and has no value at all when the variable equals a. The graph is broken.

As another example, the student should examine $f(x) = 2^{\frac{1}{x}}$ as x approaches 0 from either side.

6. Derivative. Consider the function x^2 and write

$$y = x^2.$$

Corresponding to $x = 1$ we have $y = 1$. If now we allow x to increase by Δx[1] and designate the corresponding change in y by Δy, we have

$$1 + \Delta y = (1 + \Delta x)^2 = 1 + 2\,\Delta x + (\Delta x)^2.$$

$$\therefore \Delta y = 2\,\Delta x + (\Delta x)^2. \tag{4}$$

---

[1] This notation has become classic in the Calculus. The Greek capital delta, Δ, is used to represent the difference between two values of a variable, either independent or dependent. Thus if x_1 and x_2 are two values of x, their difference $x_2 - x_1$ will be represented by Δx, read "delta x"; then $x_2 = x_1 + \Delta x$. Similarly $y_2 = y_1 + \Delta y$, where $\Delta y = y_2 - y_1$. We shall speak of Δx as an *increment in* x, and Δy as an *increment in* y. The term "increment" is used even when the change in the variable or function is negative, so that there is actually a decrement instead of an increment.

If, for example, $\Delta x = 1$, then $x + \Delta x = 2$ and the corresponding value of $y + \Delta y$ is 4; the corresponding change Δy is 3. This, of course, agrees with (4). If, instead, we let $\Delta x = 0.5$, the corresponding value of Δy is 1.25. Similarly if

$\Delta x = 0.1$, $y + \Delta y = 1.21$; whence $\Delta y = 0.21$;
$\Delta x = 0.01$, $y + \Delta y = 1.0201$; whence $\Delta y = 0.0201$;
$\Delta x = 0.001$, $y + \Delta y = 1.002001$; whence $\Delta y = 0.002001$;

and so on. Correspondingly $\frac{\Delta y}{\Delta x}$ takes the successive values 3, 2.5, 2.1, 2.01, 2.001; or, using formula (4), we have

$$\frac{\Delta y}{\Delta x} = 2 + \Delta x, \tag{5}$$

which is consistent with the values found above. Evidently (5) defines $\frac{\Delta y}{\Delta x}$ in terms of Δx, and in such a way that $\frac{\Delta y}{\Delta x}$ will differ from 2 by as small a quantity as one pleases when Δx is taken sufficiently small. Hence 2 is the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero; *i.e.* $\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = 2$. The range of values found above, starting with $x = 1$, can be tabulated as indicated.

Δx	1	0.5	0.1	0.01	0.001
Δy	3	1.25	0.21	0.0201	0.002001
$\frac{\Delta y}{\Delta x}$	3	2.5	2.1	2.01	2.001

If we start with $x = 2$, then $y = 4$. Giving x an increment Δx, we have now

$$4 + \Delta y = (2 + \Delta x)^2 = 4 + 4\,\Delta x + (\Delta x)^2;$$

whence $\Delta y = 4\,\Delta x + (\Delta x)^2$, and $\frac{\Delta y}{\Delta x} = 4 + \Delta x.$

Giving Δx the same set of values as before, we may tabulate our results thus:

Δx	1	0.5	0.1	0.01	0.001
Δy	5	2.25	0.41	0.0401	0.004001
$\dfrac{\Delta y}{\Delta x}$	5	4.5	4.1	4.01	4.001

Obviously, in this case, $\underset{\Delta x \to 0}{L} \dfrac{\Delta y}{\Delta x} = 4.$

If we now take the general case and start with $x = x_0$, any desired number, we have, letting y_0 be the corresponding value of y,

$$y_0 = x_0^2,$$

$$y_0 + \Delta y = (x_0 + \Delta x)^2 = x_0^2 + 2\,x_0\Delta x + (\Delta x)^2;$$

whence $\Delta y = 2\,x_0\Delta x + (\Delta x)^2$, and $\dfrac{\Delta y}{\Delta x} = 2\,x_0 + \Delta x.$

The smaller Δx, the more nearly will $\dfrac{\Delta y}{\Delta x}$ approach $2\,x_0$. Hence

$$\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = 2\,x_0.$$

We note that the value of this limit is given in terms of x_0; in other words, it is a function of x_0. Since x_0 is any value of x we happen to have chosen, we may say that, for $y = x^2$, corresponding to any value of x, $\underset{\Delta x \to 0}{L} \dfrac{\Delta y}{\Delta x} = 2\,x$. That is, $\underset{\Delta x \to 0}{L} \dfrac{\Delta y}{\Delta x}$ is also a function of x. It is known as the *derivative of y with respect to x.*

There are several convenient notations for the derivative. Frequently we shall use $D_x y$ or $D_x f(x)$. When it is so obvious that there is no need of emphasizing the variable with respect to which the derivative is taken, we shall use simply Dy or $Df(x)$. Or, more simply still, we shall frequently indicate the derivative by affecting the symbol for the function by a prime; thus it is convenient to represent Dy by y', and $Df(x)$ by $f'(x)$. Thus we have the following alternative forms for expressing the derivative of the function $y = x^2$,

$$D_x y = 2\,x,\ Dy = 2\,x, \text{ or } y' = 2\,x, \text{ or } D_x(x^2) = 2\,x, \text{ or } D(x^2) = 2\,x.$$

For the function $y = \frac{1}{x}$, we have, corresponding to any value $x = x_0$,

$$y_0 = \frac{1}{x_0},\ y_0 + \Delta y = \frac{1}{x_0 + \Delta x}.$$

$$\therefore\ \Delta y = \frac{1}{x_0 + \Delta x} - \frac{1}{x_0} = \frac{-\Delta x}{x_0(x_0 + \Delta x)}, \text{ and } \frac{\Delta y}{\Delta x} = \frac{-1}{x_0(x_0 + \Delta x_0)};$$

whence finally

$$\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = \frac{-1}{x_0^2}.$$

Since x_0 is any value of x, we have established that for $y = \frac{1}{x}$

$$y' = -\frac{1}{x^2}, \text{ or } D\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

For special values of x_0 we may tabulate our results thus:

$x_0 = 1$

Δx	1	0.5	0.1	0.01	0.001
Δy	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{11}$	$-\frac{1}{101}$	$-\frac{1}{1001}$
$\frac{\Delta y}{\Delta x}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{1.1}$	$-\frac{1}{1.01}$	$-\frac{1}{1.001}$

$$\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = -1$$

$x_0 = 2$

Δx	1	0.5	0.1	0.01	0.001
Δy	$-\frac{1}{6}$	$-\frac{1}{10}$	$-\frac{1}{42}$	$-\frac{1}{402}$	$-\frac{1}{4002}$
$\frac{\Delta y}{\Delta x}$	$-\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{4.2}$	$-\frac{1}{4.02}$	$-\frac{1}{4.002}$

$$\underset{\Delta x \longrightarrow 0}{L} \frac{\Delta y}{\Delta x} = -\frac{1}{4}$$

$x_0 = -3$

Δx	1	0.5	0.1	0.01	0.001
Δy	$-\frac{1}{6}$	$-\frac{1}{15}$	$-\frac{1}{87}$	$-\frac{1}{897}$	$-\frac{1}{8997}$
$\frac{\Delta y}{\Delta x}$	$-\frac{1}{6}$	$-\frac{1}{7.5}$	$-\frac{1}{8.7}$	$-\frac{1}{8.97}$	$-\frac{1}{8.997}$

$$\underset{\Delta x \longrightarrow 0}{L} \frac{\Delta y}{\Delta x} = -\frac{1}{9}$$

In general, if $f(x)$ is any function under consideration, we obtain its derivative by the method used above for the special cases where $f(x)$ was x^2 and $\frac{1}{x}$ respectively. Thus, for $x = x_0$ write

$$y_0 = f(x_0).$$

Giving x an increment Δx, so that $x = x_0 + \Delta x$, y receives an increment Δy, and we have

$$y_0 + \Delta y = f(x_0 + \Delta x).$$

Hence
$$\Delta y = f(x_0 + \Delta x) - f(x_0),$$

and
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{6}$$

The problem now is to express the right-hand member in such a form that its limit, as Δx approaches zero, can be found.[1] As the various cases arise we shall see how this is done. Passing to the limit we have

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[1] Here Δy is an infinitesimal along with Δx. For $\Delta x = 0$, $\frac{\Delta y}{\Delta x} = \frac{0}{0}$, which has no meaning. This is an illustration of the type of discontinuity referred to in 2, § 5.

$$D_x y_0 = \underset{\Delta x \to 0}{L} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = D_x f(x_0). \tag{7}$$

Since x_0 is any selected value of x, we can carry out this process, starting with an unspecified value of x. Thus

$$y = f(x).$$

$$y + \Delta y = f(x + \Delta x).$$

$$\therefore \Delta y = f(x + \Delta x) - f(x)$$

and

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x};$$

whence, passing to the limit

$$D_x y = \underset{\Delta x \to 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x} = D_x f(x). \tag{8}$$

Using (7) or (8) as the definition of the derivative of $f(x)$ corresponding to a given value of x, we see that *the derivative of a function $f(x)$, for a given value of x, is the limit of the ratio of the corresponding increment of the function to that of the variable as the latter increment approaches zero.* It also is a function of the variable, $f'(x)$, which in special cases may be a constant for all values of the variable.

EXERCISES

Find the derivatives of the following functions:

1. $y = 3x.$
2. $y = x^2 + 2x.$
3. $y = \frac{1}{2x+1}.$
4. $y = \frac{1}{1 - x^2}.$
5. $y = x^3.$
6. $y = \frac{4x}{x^2+1}.$
7. $y = \sqrt{x}.$
8. $y = \sqrt{x^2 + 1}.$

NOTE. — In the case of exercises 7 and 8 a special device is necessary to express the value of $\frac{\Delta y}{\Delta x}$ in such form as to find its limit $\Delta x \to 0$.

Thus for exercise 7,

$$y + \Delta y = \sqrt{x + \Delta x}. \qquad \therefore \Delta y = \sqrt{x + \Delta x} - \sqrt{x}.$$

Multiplying the right-hand member by $\sqrt{x + \Delta x} + \sqrt{x}$ and dividing by the same quantity, we have

$$\Delta y = \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta y} + \sqrt{x})}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

Passing to the limit $\Delta x \longrightarrow 0$,

$$Dy \equiv y' = \frac{1}{2\sqrt{x}}.$$

7. Geometrical application; slope. Starting with any point, P, on the graph of a function, $y = f(x)$, let its coördinates be (x_0, y_0). Giving x an increment Δx, y receives an increment Δy. Let Q be the new point $(x_0 + \Delta x, y_0 + \Delta y)$. Then $\frac{\Delta y}{\Delta x} = \tan QPR$, the slope of the secant line PQ.

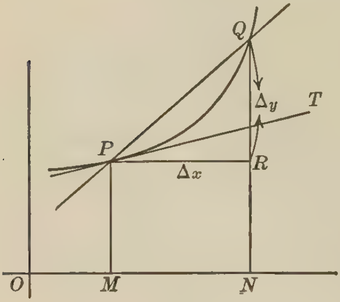

FIG. 5

As Δx is made smaller and smaller, Q approaches P more and more closely. The limiting position of the secant line, as Δx approaches zero, is PT, the tangent line at P. Hence, designating by $D_x y_0$ or y_0' the value of y' for $x = x_0$,

$$D_x y_0 \equiv y_0' = \tan RPT;$$

i.e. *in the graph of a function, the slope of the tangent at any point is equal to the value which the derivative of the function takes when the variable is equal to the abscissa of the point.*

Moreover it should be noted that when the slope (i.e. *the derivative*) *is positive, the function increases with the variable; when the slope (derivative) is negative, the function decreases as the variable increases.*

We are now in a position to write the equation of the tangent to a curve $y=f(x)$, at any point (x_0, y_0) on it, if we know the derivative of $f(x)$. It is simply a question of writing the equation of a line through the point (x_0, y_0) and with the slope y_0'. Hence *the equation of the tangent to the curve $y=f(x)$ at (x_0, y_0) is*

$$y-y_0=y_0'(x-x_0). \qquad (9)$$

Similarly *the equation of the normal to the curve at (x_0, y_0) is*

$$y-y_0=-\frac{1}{y_0'}(x-x_0), \text{ or } x-x_0+y_0'(y-y_0)=0, \qquad (10)$$

since the normal is perpendicular to the tangent, and passes through the point of contact.

For example, in the case of the parabola $y=x^2$, $y'=2x$. For $x_0=2$, $y_0=4$ and $y_0'=4$; hence the equation of the tangent at (2, 4) is $y-4=4(x-2)$, or $4x-y=4$. The equation of the normal at (2, 4) is $x-2+4(y-4)=0$, or $x+4y=18$.

For $x=x_0$, $y_0=x_0^2$ and $y_0'=2x_0$. Hence the equation of the tangent at (x_0, y_0) is $y-y_0=2x_0(x-x_0)$, or $y=2x_0x-2x_0^2+y_0$. This reduces at once to either

$$y=2x_0x-x_0^2, \text{ or } y+y_0=2x_0x.$$

The latter is the usual form given in books on Analytic Geometry.

The equation of the normal is

$$x-x_0+2x_0(y+y_0)=0, \text{ or } x+2x_0y=x_0+2x_0y_0.$$

EXERCISES

Find the equations of tangent and normal for the following curves, and sketch the curves:

1. $y = \frac{1}{x}$, at the point whose abscissa is 1; -1; 3; x_0.

2. $y = x^3$, at the point whose abscissa is 0; 1; -1; -2; x_0.

3. $y = x^2 - 2x$, at the point whose abscissa is 0; 1; 2; x_0.

4. $y = \frac{1}{1+x^2}$, at the point whose abscissa is 0; 1; -1; x_0.

8. Physical application; velocity. If, in the case of rectilinear motion (that is, motion along a straight line), the body moves a distance Δs in the interval Δt, the quotient $\frac{\Delta s}{\Delta t}$ is said to be the *average* or *mean velocity* of the body during the interval. In general the average velocity varies for different intervals. As an illustration, consider a falling body near the earth's surface. By actual experiment, or otherwise, it can be established that in the interval of time (measured in seconds) indicated in the first row of the following table, the body falls the corresponding distance (measured in feet) indicated in the second row.

Time	1	2	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{1}{8}$	$2\frac{1}{16}$	$2\frac{1}{32}$	$2\frac{1}{64}$
Distance	16	64	144	100	81	$72\frac{1}{4}$	$68\frac{1}{16}$	$66\frac{1}{64}$	$65\frac{1}{256}$

Thus, in this case, the average velocity during the
first second is 16 ft./sec. (read "16 feet per second")
first two seconds is $\frac{64}{2} = 32$ ft./sec.
first three seconds is $\frac{144}{3} = 48$ ft./sec.
third second is $\frac{144-64}{1} = 80$ ft./sec.

first half of the third second is $\frac{100 - 64}{\frac{1}{2}} = 72$ ft./sec.

first quarter of the third second is $\frac{81 - 64}{\frac{1}{4}} = 68$ ft./sec.

first eighth of the third second is $\frac{72\frac{1}{4} - 64}{\frac{1}{8}} = 66$ ft./sec.

first sixteenth of the third second is $\frac{68\frac{1}{16} - 64}{\frac{1}{16}} = 65$ ft./sec.

first thirty-second of the third second is $\frac{66\frac{1}{64} - 64}{\frac{1}{32}} = 64\frac{1}{2}$ ft./sec.

first sixty-fourth of the third second is $\frac{65\frac{1}{256} - 64}{\frac{1}{64}} = 64\frac{1}{4}$ ft./sec.

and so on. Starting the interval Δt at the end of the second second after the motion began, the above calculations show that $\frac{\Delta s}{\Delta t}$ seems to approach 64 more and more nearly as Δt grows smaller and smaller. Of course, there are physical limitations to carrying on the measurement of distances and intervals of time beyond certain stages.

Whenever it is possible to give by a formula the distance covered by a body, moving in a straight line, in any time t, it is possible to find out whether $\frac{\Delta s}{\Delta t}$ has a limit as Δt approaches zero, where Δt is an interval of time beginning at the end of the t-th second. $\underset{\Delta t \longrightarrow 0}{L} \frac{\Delta s}{\Delta t}$ is nothing but $D_t s$, the derivative of s with respect to t.

The limit of the average velocity as the interval of time Δt approaches zero is defined as the *instantaneous velocity*, or simply the *velocity* of the body at the instant when the ultimately vanishing interval of time begins. Thus if $s = f(t)$, $D_t s_0 \equiv f'(t_0)$ represents the velocity of the body at the instant t_0.

In the case of the falling body

$$s = 16\,t^2.$$

It will be left as an exercise to show that in this case $D_ts = 32\,t$. For $t = 2$, this becomes 64, as we anticipated above.

If the average velocity of a body is the same for all intervals of time (no matter what their lengths or at what instants they begin), the velocity is said to be *uniform*. Thus if a body is moving with uniform velocity, $\frac{\Delta s}{\Delta t} = k$, a constant for all values of Δt. Hence $D_ts = k$; that is, *the velocity is constant in the case of uniform rectilinear motion.*

It may help to realize what is meant by velocity by noting that if beginning with some instant t_0 the motion were suddenly to become uniform with the velocity v_0 at that instant, the distance traveled by the body in the next unit of time would be equal to v_0.

EXERCISES

Find the expressions for the velocity in the following cases of rectilinear motion:

1. $s = 16\,t^2 + 10\,t$.
2. $s = t^3 - 3\,t^2$.
3. $s = \frac{2}{t}$.
4. $s = t - \frac{1}{t}$.

CHAPTER II

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

9. Differentiation formulas. We saw in §§ 7 and 8 that the derivative has very important geometrical and physical interpretations. Later we shall see further applications of the derivative. While skill in the use of the general directions for finding the derivative of a function given in § 6 may enable the student to differentiate[1] a large number of functions he is apt to encounter, it is found in practice that much time and labor can be saved by deducing general differentiation formulas and then *memorizing them for use.*

10. Derivative of a polynomial.

1. If $y = c$, where c is a constant,

$y + \Delta y = c. \quad \therefore \Delta y = 0$, and $\dfrac{\Delta y}{\Delta x} = 0. \quad \therefore \underset{\Delta x \to 0}{L} \dfrac{\Delta y}{\Delta x} = 0.$ Hence

$$\textbf{I.} \qquad D_x c = 0;$$

i.e. *the derivative of a constant is zero.*

2. If $y = x$, $y + \Delta y = x + \Delta x. \quad \therefore \Delta y = \Delta x$ and $\dfrac{\Delta y}{\Delta x} = 1.$

$\therefore \underset{\Delta x \to 0}{L} \dfrac{\Delta y}{\Delta x} = 1.$ Hence

$$\textbf{II.} \qquad D_x x = 1;$$

i.e. *the derivative of a variable with respect to itself is unity.*

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[1] The process of "finding the derivative of a function" is known as "differentiating it."

3. If $y = x^n$, where n is a positive integer,

$$y + \Delta y = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2\,!}x^{n-2}(\Delta x)^2 + \cdots$$
$$+ nx(\Delta x)^{n-1} + (\Delta x)^n.$$

Subtracting the first equation from the second and dividing by Δx,

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2\,!}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}.$$

Each of the terms on the right, except the first, contains a positive power of Δx as a factor. The limit of each of these terms is zero as Δx approaches zero. Passing to the limit,[1]

$$\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = nx^{n-1}. \quad \text{Hence}$$

III. $$\boldsymbol{D_x x^n = nx^{n-1}};$$

i.e. *the derivative of the nth power of a variable is n times the* $(n-1)$*th power of the variable.*

Remark. — This formula was proved above for n, a positive integer. But it holds equally well, as we shall see below, even when n is *any* constant number.

4. If $y = cf(x)$, where c is any constant,

$$y + \Delta y = cf(x + \Delta x). \quad \therefore \Delta y = c[f(x + \Delta x) - f(x)],$$

and $$\frac{\Delta y}{\Delta x} = c\frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad \therefore \underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = cD_x f(x). \quad \text{Hence}$$

IV. $$\boldsymbol{D_x[cf(x)] = cD_x f(x) = cf'(x)};$$

i.e. *the derivative of a constant times a function is equal to the constant times the derivative of the function.*

Thus $$D_x(3x^2) = 3D_x x^2 = 6x.$$

Also $$D_x\left(\frac{x^3}{6}\right) = \frac{1}{6}D_x x^3 = \frac{3x^2}{6} = \frac{x^2}{2}.$$

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[1] Here we are using a general principle in the theory of limits that "the limit of the algebraic sum of two or more functions is equal to the algebraic sum of their limits."

5. If $y = f(x) \pm \phi(x)$,

$$y + \Delta y = f(x + \Delta x) \pm \phi(x + \Delta x),$$

and
$$\Delta y = f(x + \Delta x) - f(x) \pm [\phi(x + \Delta x) - \phi(x)].$$

$$\therefore \ \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \pm \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Passing to the limit,

$$\underset{\Delta x \to 0}{L} \frac{\Delta y}{\Delta x} = D_x f(x) \pm D_x \phi(x). \quad \text{Hence}$$

$$\textbf{V.} \quad D_x[f(x) \pm \phi(x)] = D_x f(x) \pm D_x \phi(x) = f'(x) \pm \phi'(x);$$

i.e. *the derivative of the sum (or difference) of two functions is equal to the sum (or difference) of the derivatives of the functions.*

In exactly the same way one can extend the formula to the algebraic sum of any finite number of functions, thus

$$\textbf{V}'. \quad D_x[f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]$$
$$= D_x f_1(x) \pm D_x f_2(x) \pm \cdots \pm D_x f_n(x) = f_1'(x) \pm f_2'(x) \pm \cdots \pm f_n'(x);$$

i.e. *the derivative of the algebraic sum of any finite number of functions is the sum of the derivatives of those functions.*

Remark. — By means of these five formulas we can differentiate any polynomial very readily. By a *polynomial* in a variable is meant a *rational entire* function of the variable; that is, a function given by an expression in which the only operations upon the variable are those of addition, subtraction, and multiplication. A polynomial is then simply a sum of terms, each of which involves the variable to a positive integral degree. The number of terms entering into the expression plays no rôle, neither does the nature of the coefficients. Thus x^2, $3x - \sqrt{2}$, $x^2 \cos a - x \sin a + 1$, 3 (which may be looked upon as $3x^0$) are all polynomials in x.

A *rational* function is a more general type than polynomial. It permits the four operations of addition, subtraction, multiplication, and division. It includes the polynomial as a special case. In the general case, however, we have a *rational fraction*, which may always be expressed as the quotient of two polynomials.

Thus, if $y = x^5 - 3x^4 + 2x^2 + 5x - 7,$

$$D_xy = D_xx^5 - 3D_xx^4 + 2D_xx^2 + 5D_xx - D_x7$$
$$= 5x^4 - 12x^3 + 4x + 5.$$

Again, if $y = ax^m + bx^n + cx^p,$

$$y' = amx^{m-1} + bnx^{n-1} + cpx^{p-1}.$$

And if $y = 3u^3 + 6u - 5,$

$$D_uy = 9u^2 + 6 = 3(u^2 + 2).$$

EXERCISES

Find the derivatives of

1. $x^3 - 9x^2 + 8.$ 2. $5u^2 - 4u - 2.$ 3. $3t^2 - t^3.$

Find the equations of tangents and normals to the following curves at the various points indicated:

4. $2y = x^2 - 2x + 4$, at the point whose abscissa is 0; 2; -3.

5. $y = x^3 - x$, at the point whose abscissa is 0; 1; -1.

6. $y = x^4 - 6x^2 - 1$, at the point whose abscissa is 0; -2; 2.

Find the velocity in the cases of rectilinear motion given by the following equations:

7. $s = 3t^4 - 4t^3 + 6t$, when $t = 0$; 1; 2; 3.

8. $s = 2t$, when $t = 0$; 2; 10.

9. $s = 16t^2 - 6t + 1$, when $t = 0$; 3; 4.

11. Derivative of a function of a function. Frequently a function of a variable assumes a simpler form when expressed in terms of an auxiliary variable, itself a function of the original variable. Thus $(x^3 + 2)^2$ becomes u^2 if $x^3 + 2 = u$,

$$\sin\sqrt{1 + x^2} \text{ becomes } \sin u \text{ if } \sqrt{1 + x^2} = u,$$

and so on.

The process of differentiating a function is also frequently simplified by the introduction of an auxiliary variable. This results from the following principle, which may seem somewhat formidable in its general statement, but is exceedingly simple in its application.

Suppose we have simplified the form of a function of x by introducing an auxiliary variable u, itself a continuous function of x, thus having

$$y = f(u).$$

Giving x the increment Δx, y receives an increment Δy, and u receives an increment Δu; *i.e.*

$$y + \Delta y = f(u + \Delta u) \quad \therefore \Delta y = f(u + \Delta u) - f(u), \text{ and}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta x}.$$

In order to be able to find the limit here, we should have Δu in the denominator on the right instead of Δx. Compare (8), § 6. This may be effected by multiplying numerator and denominator of the right-hand member by Δu, and arranging thus

$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

As Δx approaches zero, so also does Δu. Passing to the limit,[1]

VI. $$D_x y = D_u f(u) D_x u \quad \text{or} \quad D_x f(u) = D_u f(u) D_x u;$$

i.e. *the derivative of a function of a function of a variable with respect to the variable is equal to the derivative of the original function with respect to the auxiliary function, multiplied by the derivative of the latter with respect to the variable.*

This formula is of prime importance, and we shall have frequent occasion to use it.

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[1] Here we are using a general principle in the theory of limits: "the limit of the product of two functions is equal to the product of their limits," thus

$$\underset{x \to a}{L} [F(x) \cdot \Phi(x)] = \underset{x \to a}{L} F(x) \cdot \underset{x \to a}{L} \Phi(x).$$

1. As a first application consider $y = u^n$. Using VI, we have

$$D_x y = D_u u^n D_x u = nu^{n-1} D_x u, \quad \text{or} \quad y' = nu^{n-1}u'.$$

Here it is understood that the prime indicates the derivative with respect to x.

This important generalization of III is worth noting and memorizing.

III′. $$D_x u^n = nu^{n-1} D_x u.$$

As an example consider $y = (x^3 + 6x - 1)^5$. Instead of expanding this as a polynomial of the 15th degree in x and then differentiating term for term, we shall let $x^3 + 6x - 1 = u$, whence $y = u^5$, and

$$y' = 5u^4 u' = 5(x^3 + 6x - 1)^4(3x^2 + 6) = 15(x^3 + 6x - 1)^4(x^2 + 2).$$

It is preferable not to expand this result, but to keep it in the compact form in which we have obtained it.

In an analogous manner the student should find the derivatives of the following:

1. $(x^2 - 2x + 3)^3$.

2. $(x^5 - 1)^4$.

3. $(x^4 - 3x^2 + 6x)^3$.

4. $(x^2 + 2x - 1)^2$.

2. In order to make a second important application of formula VI we shall introduce the notion of "inverse functions." If y is a function of x, then, in general, x is a function of y; and each of these two functions is said to be the *inverse* of the other. Thus if $y = x^2$, then $x = \sqrt{y}$; that is, the *square* and the *square root* are inverse functions. Again, if $y = \sin x$, $x = \sin^{-1} y$; that is, *sine* and *arcsine* are inverse functions.

If $y = f(x)$ and $x = \phi(y)$, the functions f and ϕ are inverse functions. Evidently, from their definition, $\phi[f(x)] = x$. Writing this $\phi(y) = x$, we have, on differentiating with respect to x and using VI,

$$D_x\phi(y) = D_y\,\phi(y) D_x y = 1, \quad \text{or} \quad D_y x\, D_x y = 1, \quad \text{or}$$

VII.
$$D_y x = \frac{1}{D_x y};$$

i.e. *the derivative of x with respect to y is equal to the reciprocal of the derivative of y with respect to x.*

3. We are now in a position to show that formula III holds also for n any positive fraction $\frac{p}{q}$, where p and q are integers. First consider $u = x^{\frac{1}{q}}$; whence $x = u^q$, and, using III,

$$D_u x = qu^{q-1} = qx^{1-\frac{1}{q}}, \text{ or, using VII,}$$

$$D_x u = \frac{1}{q} x^{\frac{1}{q}-1}.$$

Now, if $y = x^{\frac{p}{q}}$, letting $x^{\frac{1}{q}} = u$, we have

$$y = u^p; \text{ whence, using VI or III}',$$

$$D_x y = pu^{p-1} D_x u = pu^{p-1} \frac{1}{q} x^{\frac{1}{q}-1} = \frac{p}{q} x^{\frac{p-1}{q}+\frac{1}{q}-1}$$
$$= \frac{p}{q} x^{\frac{p}{q}-1}.$$

This formula is of very wide application. Thus, consider $y = \sqrt{x^2 - 1}$. Letting $x^2 - 1 = u$, we may write $y = u^{\frac{1}{2}}$; whence

$$D_x y = \tfrac{1}{2} u^{-\frac{1}{2}} D_x u = \frac{D_x u'}{2u^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2-1}}.$$

Again, take $y = \sqrt[3]{(1-x)^2}$. Letting $1 - x = u$, we have

$$y = u^{\frac{2}{3}}. \quad \therefore D_x y = \tfrac{2}{3} u^{-\frac{1}{3}} D_x u = \frac{2}{3u^{\frac{1}{3}}}(-1) = \frac{-2}{3\sqrt[3]{1-x}}.$$

REMARK. — The problem of differentiating the square root of a function occurs so frequently in practice that it may be well to remember that

III″.
$$D_x \sqrt{u} = \frac{D_x u}{2\sqrt{u}} = \frac{u'}{2\sqrt{u}},$$

or, in words, *the derivative of the square root of a function is equal to the derivative of the function divided by twice the square root of the function.*

Differentiate the following:

5. $\sqrt{1-2x+x^3}$.
7. $\sqrt{2t-1}$.
6. $(1-2x+2x^2)^{\frac{3}{2}}$.
8. $\sqrt{(x-a)^2+b^2}$.

Find the equations of the tangents to the following curves at the points indicated:

9. $y^2=8x+9$ at the points whose abscissas are 0; 2; -1.

10. $x^2+y^2=25$ at the points whose abscissas are 0; -3; 4; 5.

11. $2x^2-y^2=1$ at the points whose abscissas are 1; -5.

12. $y^3=x^2$ at the points whose abscissas are 1; 8; 0.

12. Derivative of a product. If u and v are given functions of x, when x is changed to $x+\Delta x$, u and v become $u+\Delta u$ and $v+\Delta v$, respectively. Hence, starting with

$y=uv$, we have

$$y+\Delta y=(u+\Delta u)(v+\Delta v)=uv+v\,\Delta u+u\,\Delta v+\Delta u\,\Delta v.$$

$$\therefore \Delta y=v\,\Delta u+u\,\Delta v+\Delta u\,\Delta v, \text{ and}$$

$$\frac{\Delta y}{\Delta x}=v\frac{\Delta u}{\Delta x}+u\frac{\Delta v}{\Delta x}+\Delta u\frac{\Delta v}{\Delta x}.$$

As Δx approaches 0, Δu and Δv each also approaches 0, and $\frac{\Delta u}{\Delta x}$ and $\frac{\Delta v}{\Delta x}$ approach D_xu and D_xv, respectively, by the definition of a derivative. Hence[1] $\underset{\Delta x \to 0}{L}\ \Delta u\frac{\Delta v}{\Delta x}=0\cdot D_xv=0$, and

VIII. $$D_xuv=v\,D_xu+u\,D_xv=vu'+uv',$$

i.e. *the derivative of the product of two functions is equal to the second function times the derivative of the first plus the first function times the derivative of the second.*

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[1] Here we again make use of the principle that "the limit of the product is equal to the product of the limits"

Thus, if $y = x^2\sqrt{1-x^2}$,

$$Dy = \sqrt{1-x^2}\,Dx^2 + x^2\,D\sqrt{1-x^2}$$

$$= 2x\sqrt{1-x^2} + x^2\frac{-x}{\sqrt{1-x^2}} = 2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}}.$$

While this is a possible form in which to leave the derivative, it is preferable to reduce to a common denominator and simplify the resulting fraction; thus

$$Dy = \frac{2x(1-x^2)-x^3}{\sqrt{1-x^2}} = \frac{2x-3x^3}{\sqrt{1-x^2}}.$$

Differentiate the following functions:

1. $x(1-x)^3$.
2. $x^2(a+bx)^4$.
3. $x\sqrt{x^2-x+1}$.
4. $(x^2+2)\sqrt[3]{x^2-1}$.
5. $x^3\sqrt{1-x^2}$.
6. $x^{\frac{2}{3}}(x^2-4)^{\frac{1}{3}}$.

Ex. Show that formula IV is a special case of VIII.

Remark. — Formula VIII can be extended to apply to the product of any finite number of functions.

Thus if $y = u_1 u_2 u_3$, let $u_2 u_3 = v$.

Then $y = u_1 v$ and $D_x y = v\,D_x u_1 + u_1 D_x v$; whence,

$$D_x y = u_2 u_3 D_x u_1 + u_1 u_3 D_x u_2 + u_1 u_2 D_x u_3.$$

Step by step this can be extended to the product of n functions, giving

VIII′. $$D_x u_1 u_2 u_3 \cdots u_n = u_2 u_3 \cdots u_n D_x u_1 + u_1 u_3 u_4 \cdots u_n D_x u_2 + \cdots + u_1 u_2 \cdots u_{n-1} D_x u_n;$$

i.e. *the derivative of a product of functions is equal to the sum of terms obtained by multiplying the derivative of each function by the product of all the remaining functions.*

Ex. Deduce III′, § 11, from VIII′ by putting

$$u_1 = u_2 = \cdots = u_n = u.$$

13. Derivative of a quotient. Starting with $y = \frac{u}{v}$ and giving x an increment Δx,

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}; \text{ whence}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)}, \text{ and}$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$ Passing to the limit[1] $\Delta x \rightarrow 0$,

$$\text{IX.} \qquad D_x\frac{u}{v} = \frac{v\,D_x u - u\,D_x v}{v^2} = \frac{vu' - uv'}{v^2};$$

i.e. *the derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Thus if $$y = \frac{x^2}{\sqrt{1 - x^2}},$$

$$y' = \frac{\sqrt{1 - x^2}\,Dx^2 - x^2 D\sqrt{1 - x^2}}{1 - x^2}$$

$$= \frac{2x\sqrt{1 - x^2} + \dfrac{x^3}{\sqrt{1 - x^2}}}{1 - x^2}.$$

Multiplying numerator and denominator by $\sqrt{1 - x^2}$,

$$y' = \frac{2x(1 - x^2) + x^3}{(1 - x^2)\sqrt{1 - x^2}} = \frac{2x - x^3}{(1 - x^2)^{\frac{3}{2}}}.$$

Again, if $y = \dfrac{\sqrt{1 - t}}{\sqrt{1 + t}}$,

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[1] Here we are making use of the additional principle that "the limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the function in the denominator is different from zero."

$$D_t y = \frac{\sqrt{1+t}\,D_t\sqrt{1-t} - \sqrt{1-t}\,D_t\sqrt{1+t}}{1+t}$$

$$= \frac{\dfrac{-\sqrt{1+t}}{2\sqrt{1-t}} - \dfrac{\sqrt{1-t}}{2\sqrt{1+t}}}{1+t}.$$

The least common denominator of the fractions in the numerator is $2\sqrt{1-t}\sqrt{1+t}$. Multiplying numerator and denominator by this,

$$D_t y = \frac{-(1+t)-(1-t)}{2\sqrt{(1+t)^3(1-t)}} = \frac{-1}{\sqrt{(1+t)^3(1-t)}}.$$

This function can be differentiated also by letting $\frac{1-t}{1+t} = u$ and using III′ on $y = u^{\frac{1}{2}}$. It is suggested, as an exercise, that the student actually do this and compare his work with that given in the text.

Differentiate the following functions:

1. $\dfrac{a^2+x^2}{a^2-x^2}$.

2. $\dfrac{t}{\sqrt{a-bt^2}}$.

3. $\dfrac{x^2-1}{\sqrt{x^2+1}}$.

4. $\dfrac{r^{\frac{2}{3}}}{(r^2-1)^{\frac{1}{3}}}$.

We are now in a position to show that formula III (or III′) holds also when the exponent is negative.

Thus if

$$y = u^{-m} = \frac{1}{u^m}, \text{ we have from IX}$$

$$D_x y = \frac{-D_x u^m}{(u^m)^2} = \frac{-m\,u^{m-1}D_x u}{u^{2m}} = -m\,u^{-m-1}D_x u.$$

Formula III (or its generalized form III′) has now been shown to hold for any rational value of n, either positive or negative.

Note 1. — In case the numerator of a fraction to be differentiated is a constant, thus if $y = \frac{c}{v}$, write $x = cv^{-1}$ and use III′ instead of IX. For example, if $y = \frac{3}{(x^2 - 3x + 2)^2}$, write $y = 3(x^2 - 3x + 2)^{-2}$; whence

$$y' = -6(x^2 - 3x + 2)^{-3}(2x - 3) = \frac{18 - 12x}{(x^2 - 3x + 2)^3}.$$

Note 2. — If the denominator is a constant, IV should be used instead of IX.

Thus if $y = \frac{u}{c}$, write $y = \frac{1}{c}u$; then $y' = \frac{1}{c}u' = \frac{u'}{c}$.

Note 3. — Notes 1 and 2 may be included in the more general statement that whenever the function to be differentiated contains a constant factor it is advisable to bring this fact into evidence and use IV. Thus if $v = \sqrt{2gs}$, where g is a constant, write it $v = \sqrt{2g}\sqrt{s}$, whence

$$D_s v = \sqrt{2g}\, D_s\sqrt{s} = \frac{\sqrt{2g}}{2\sqrt{s}} = \sqrt{\frac{g}{2s}}.$$

Note 4. — Use should also be made of III instead of IX in case the denominator is a monomial.

Thus, if $y = \frac{x^2 - 3x + 2}{6x}$, write it $y = \frac{x}{6} - \frac{1}{2} + \frac{x^{-1}}{3}$; whence

$$y' = \frac{1}{6} - \frac{x^{-2}}{3} = \frac{x^2 - 2}{6x^2}.$$

This principle can be extended to the case where

$$y = \frac{au^n + bu^{n-1} + \cdots + lu^r}{u^k},$$

u being any function of x. Write this $y = au^{n-k} + bu^{n-k-1} + \cdots + lu^{r-k}$, and use III′. As a simple example, let $y = \frac{x - 1}{x + 1}$.

Write $y = \frac{x + 1 - 2}{x + 1} = 1 - \frac{2}{x + 1}$. Then $y' = \frac{2}{(x + 1)^2}$.

Note 5. — Occasionally it is of practical advantage to extend the principle involved in Note 1 by writing $\frac{u}{v}$ as uv^{-1} and then differentiating this as a product. This applies especially to the case when v contains a factor to a higher power than the first. According to the remark below, this factor will appear to a degree one higher in the denominator of the derivative; while formula IX applied to $\frac{u}{v}$ gives v^2 in the denominator, which means that a common factor in numerator and denominator must still be gotten rid of. On the other hand, differentiating the product uv^{-1} will bring each factor to its proper degree at once.

NOTE 6. — In the case of certain fractions, the problem of differentiation may be simplified by adding and subtracting the same term in the numerator, thus enabling one to replace the function by the algebraic sum of functions, each of which is easy to differentiate. Thus in the case of Ex. 1, on adding and subtracting a^2 in the numerator, the fraction may be written $\frac{2a^2}{a^2 - x^2} - 1$; again, in the case of the illustrative exercise, $\frac{x^2}{\sqrt{1-x^2}}$, adding and subtracting 1 in the numerator, this fraction is seen to equal $\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}$.

REMARK. — *If a function contains a factor k times* (where k may be integral or fractional, positive or negative), *its derivative will contain that factor* $k - 1$ *times.* If $k = 1$, the factor does not appear in the derivative as a factor. Thus if the function contains the factor u k times, it may be written $y = u^k v$. Then $y' = v \cdot ku^{k-1}u' + u^k v' = u^{k-1}(kvu' + uv')$.

This fact is of practical importance as a check, and is also of service in the sketching of curves.

As illustrations of this fact, we recall that we found

$$D_x \frac{x^2}{\sqrt{1-x^2}} = \frac{x(2-x^2)}{(1-x)^{\frac{3}{2}}};$$

here x appears to the power $2 - 1$, and $(1 - x)$ to the power $-\frac{1}{2} - 1$.

Also $D_t \frac{\sqrt{1-t}}{\sqrt{1+t}} = \frac{-1}{(1-t)^{\frac{1}{2}}(1+t)^{\frac{3}{2}}}$, where $(1 - t)$ appears to the power $\frac{1}{2} - 1$, and $(1 + t)$ to the power $-\frac{1}{2} - 1$.

The student is advised to accustom himself to applying this check.

Differentiate the following functions:

5. $\frac{x^2-1}{x^2+1}$.

6. $\frac{1}{(1-x^2)^3}$.

7. $\frac{x}{x^4+1}$.

8. $\frac{(x^2+1)^3}{x^3}$.

9. $\frac{4x+9}{(x+1)^2}$.

10. $\frac{x}{\sqrt{1+x^2}}$.

11. $\sqrt{\frac{x}{1+x^2}}$.

12. $\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

HINT. — Rationalize the denominator in Ex. 12 before differentiating.

14. Summary. The general formulas of differentiation deduced in this chapter will enable one to differentiate any algebraic[1] function of one variable. It is not possible to overestimate the importance of the student's ability to use them. For purposes of reference, they are given below, but it is highly essential that the student memorize them. The numerous exercises that follow are intended not only to afford practice in the use of these formulas, but also to recall some of the elementary principles in the reduction of algebraic expressions to simplest possible forms. A reasonable amount of attention to this at this stage will be found very serviceable later on.

I. $D_x c = 0$.

II. $D_x x = 1$.

$$\left.\begin{aligned} &\text{III.} \quad D_x x^n = nx^{n-1} \\ &\text{III}'. \quad D_x u^n = nu^{n-1} D^x u \end{aligned}\right\}. \quad \textbf{These hold for all values of } n.$$

---

[1] The term "algebraic function" is used here in a restricted sense to mean a function of the variable that is obtained by applying to the variable any of the operations of addition, subtraction, multiplication, division, and extraction of roots.

III″. $D_x\sqrt{u} = \dfrac{D_x u}{2\sqrt{u}}$.

IV. $D_x cu = c\,D_x u$.

V′. $D_x(u_1 + u_2 + \cdots + u_n) = D_x u_1 + D_x u_2 + \cdots + D_x u_r$.

VI. $D_x f(u) = D_u f(u)\,D_x u$.

VII. $D_y x = \dfrac{1}{D_x y}$.

VIII. $D_x(uv) = v\,D_x u + u\,D_x v$.

IX. $D_x\left(\dfrac{u}{v}\right) = \dfrac{v\,D_x u - u\,D_x v}{v^2}$.

NOTE 1. — If $y = \dfrac{c}{v}$, write $y = cv^{-1}$ and use III′ instead of IX.

NOTE 2. — If $y = \dfrac{u}{c}$, write $y = \dfrac{1}{c}u$ and use IV.

NOTE 4. If $y = \dfrac{au^n + bu^{n-1} + cu^{n-2} + \cdots}{u^k}$,
write $y = au^{n-k} + bu^{n-k-1} + cu^{n-k-2} + \cdots$, and use III′.

In general it is desirable to examine the function to be differentiated to see whether it can be reduced to a simpler form before starting to differentiate. Compare Note 6 and Ex. 12, § 13.

EXERCISES

Differentiate the following functions:

1. $\dfrac{x+1}{\sqrt{x}}$.
2. $x(2x-1)(3x+2)$.
3. $\dfrac{1-2x}{(x-1)^2}$.
4. $\dfrac{1}{\sqrt{1-x^2}}$.
5. $(x-1)^2(x+2)$.
6. $\dfrac{(\sqrt{x}-1)^2}{x}$.
7. $\dfrac{x}{(x-1)^2}$.
8. $\dfrac{x^3}{(1+x)^2}$.
9. $\dfrac{(1+x)^2}{x^3}$.

10. $(x^2 - 1)\sqrt{1 - x^2}$.

11. $\dfrac{1 + x^2}{\sqrt{1 + x^2}}$.

12. $\dfrac{(1 + x)^2}{\sqrt{1 + x^2}}$.

13. $\dfrac{a^2 + x^2}{\sqrt{a^2 - x^2}}$.

14. $(a^2 + x^2)\sqrt{a^2 - x^2}$.

15. $(\sqrt{a} - \sqrt{x})^2$.

16. $\dfrac{x^2 + 1}{x^2 - 1}$.

17. $\dfrac{\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2}{x}$.

18. $\sqrt{f^2(x) + \phi^2(x)}$.

19. $\dfrac{1}{\sqrt{x^2 + f^2(x)}}$.

20. $\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$.

21. $\sqrt[3]{\dfrac{1 - x}{1 + x}}$.

22. $\dfrac{x}{\sqrt{x^2 - a^2}}$.

23. $x\sqrt{x^2 - a^2}$.

24. $\dfrac{\sqrt{x^2 - a^2}}{x}$.

25. $\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

26. $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)(x - 1)$.

27. $\dfrac{x}{(1 + \sqrt{x})^2}$.

28. $(a^{\frac{2}{3}} - x^{\frac{2}{3}})^3$.

29. $(a + x)\sqrt{a - x}$.

30. $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$.

31. $\dfrac{x + 3}{(x + 2)^2}$.

32. $\sqrt{\dfrac{x}{1 - x^2}}$.

33. $\sqrt{x}\left(x + \dfrac{1}{x}\right)^2$.

34. $\dfrac{2x^2 - 1}{3x^3}\sqrt{x^2 - 1}$.

35. $\dfrac{x}{1 + \sqrt{1 - x^2}}$.

15. Second and higher derivatives. We have seen that the derivative of a function of a variable is again a function of that variable. Thus

if $f(x) = 3x^2 - 2x + 1$, $\quad f'(x) = 6x - 2$;

if $\phi(x) = \sqrt{1 + x^2}$, $\quad \phi'(x) = \dfrac{x}{\sqrt{1 + x^2}}$.

Frequently it is important to consider the derivative of the derivative. When this exists, it is called the *second derivative* of the original function, and is designated by a double prime or by prefixing D_x^2 to the function symbol. Thus since

$$D_x f'(x) = 6,$$

we have
$$f''(x) = D_x^2 f(x) = 6,$$

and since
$$D_x \phi'(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}},$$

we have
$$\phi''(x) = D_x^2 \phi(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

Similarly the derivative of the second derivative is called the *third derivative* of the original function, and is designated by a triple prime or by prefixing D_x^3 to the function symbol. Thus $f'''(x) = D_x^3 f(x) = 0$, and $\phi'''(x) = D_x^3 \phi(x) = \dfrac{-3x}{(1+x^2)^{\frac{5}{2}}}$. And so on.

Find the first four successive derivatives of the following functions:

1. $x^4 - 2x^3 + 6$.
2. $x^5 - 3x^4 + x + 4$.
3. $(x^2 + 4x - 1)^2$.
4. $(x^3 - 3x + 2)^2$.
5. $\dfrac{1}{1-x}$.
6. $\sqrt{x}$.
7. $\dfrac{x}{1-x}$.
8. $x^{\frac{1}{3}} - x^{-\frac{1}{3}}$.
9. $\sqrt{x^2 + a^2}$.

Formula VI, § 11, plays a most important rôle when looked upon as a formula for *change of variable.* It gives the value of $D_x y$, the derivative of y with respect to x, in terms of $D_z y$, the derivative of y with respect to a new variable z, and $D_x z$, the derivative of this new variable with respect to x. Here z may be either a new dependent variable or a new independent one.

Repeated application of formula VI enables one to effect a change of variable in higher derivatives also. Thus introducing the new variable z, dependent or independent,

$$\begin{aligned}
D_x y &= D_z y\, D_x z.\\
D_x^2 y &= D_x(D_z y\, D_x z) = D_x(D_z y) D_x z + D_z y\, D_x(D_x z)\\
&= D_z^2 y (D_x z)^2 + D_z y\, D_x^2 z.\\
D_x^3 y &= D_x(D_x^2 y)\\
&= D_z^3 y (D_x z)^3 + 3\, D_z^2 y\, D_x^2 z\, D_x z + D_z y\, D_x^3 z.
\end{aligned}$$

If z is a new independent variable and x is given in terms of z, we have the alternative expressions for the formulas just given, making use of the fact that $D_x z = \dfrac{1}{D_z x}$,

$$D_x y = \frac{D_z y}{D_z x}.$$

$$D_x^2 y = D_x\left(\frac{D_z y}{D_z x}\right) = \frac{D_z\left(\dfrac{D_z y}{D_z x}\right)}{D_z x} = \frac{D_z^2 y\, D_z x - D_z y\, D_z^2 x}{(D_z x)^3}.$$

$$\begin{aligned}
D_x^3 y &= D_x(D_x^2 y) = \frac{D_z(D_x^2 y)}{D_z x}\\
&= \frac{D_z^3 y (D_z x)^2 - 3\, D_z^2 y\, D_z^2 x\, D_z x - D_z y\, D_z^3 x + 3\, D_z y (D_z^2 x)^2}{(D_z x)^5}.
\end{aligned}$$

In particular if $z = y$,

$$\begin{aligned}
D_x y &= \frac{1}{D_y x}.\\
D_x^2 y &= -\frac{D_y^2 x}{(D_y x)^3}.\\
D_x^3 y &= \frac{3(D_y^2 x)^2 - D_y^3 x\, D_y x}{(D_y x)^5}.
\end{aligned}$$

In the following exercises use the formulas of transformation given at the top of the page.

10. Transform $y = 2\,x\, D_x y + y^2 (D_x y)^3$ by letting $y = \sqrt{z}$.

11. Transform $x^2 y\, D_x^2 y + (x D_x y - y)^2 = 0$ by letting $y = \sqrt{z}$.

12. Transform $D_x^2 y - \dfrac{1}{x} D_x y + 4\, x^2 y = 0$ by letting $z = x^2$.

16. Implicit functions. If y is not given directly as a function of x, but is defined by an equation involving the two variables, it is said to be *given implicitly* as a function of x, or it is said to be an *implicit function* of x. If the equation is solved for y, each of its values is said to be an *explicit function* of x. Thus $x^2 + y^2 = 25$ defines y as an implicit function of x. From this relation we can find y to be either of the two explicit functions $+\sqrt{25 - x^2}$, $-\sqrt{25 - x^2}$. Again, $x^2 + 2xy + y^2 - 5x + 4 = 0$ defines y as an implicit function. Solving, we have y given explicitly as $-x \pm \sqrt{5x - 4}$.

It is possible and frequently preferable to find the derivative of an implicit function as a function of both variables directly, instead of first finding the function explicitly. The rule is to differentiate each member of the equation defining the function, remembering that y is a function of x, and then equating the derivatives of the two members. Thus starting with $x^2 + y^2 = 25$, we have, differentiating each member and equating results,

$$2x + 2y\,Dy = 0; \text{ whence } Dy = -\frac{x}{y}.$$

If desired, y may now be replaced by its value in terms of x obtained from the original equation. But it is frequently simpler to leave the derivative as we find it. Thus if the problem is to find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$, we see at once that the slope of the tangent at that point is $\frac{3}{4}$, and hence its equation is $3x - 4y = 25$.

Again, if we are asked to find the equation of the tangent to the parabola $x^2 + 2xy + y^2 - 5x + 4 = 0$ at the point $(8, -2)$, we have, differentiating each term and using y' instead of Dy,

$$2x + 2y + 2xy' + 2yy' - 5 = 0. \quad \therefore y' = \frac{5 - 2x - 2y}{2x + 2y}.$$

For $x = 8$, $y = -2$, $y' = -\frac{7}{12}$. Hence the equation of the tangent at $(8, -2)$ is $7x + 12y = 32$.

EXERCISES

Find the equations of the tangents to the following curves:

1. $x^2 + 3y^2 = 4$ at $(1, -1)$; (x_0, y_0).
2. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$ at $(4, 9)$; (x_0, y_0).
3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$ at $(8, -1)$; (x_0, y_0).
4. $x^n + y^n = a^n$ at (x_0, y_0).
5. $x^2 + xy - y^2 = 5$ at $(3, -1)$; (x_0, y_0).
6. $y^3 + 3x^2y + x^2 - 2xy - 3 = 0$ at $(1, 1)$.
7. $y^2 = 4ax$ at $(a, 2a)$; (x_0, y_0).
8. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_0, y_0).

17. Higher derivatives of implicit functions. If a function is given implicitly, we may express its derivative in the fractional form given in the previous section. Thus we saw that if y is defined by the relation $x^2 + y^2 = 25$, $y' = -\frac{x}{y}$; also that if it is defined by $x^2 + 2xy + y^2 - 5x + 4 = 0$, $y' = \frac{5 - 2x - 2y}{2x + 2y}$.

The second derivative may then be obtained by differentiating the right-hand member. This will involve y', the first derivative. Replacing this by its value, already found, in terms of x and y, we have the second derivative in terms of x and y only. The process may then be extended to the third and higher derivatives. In the case of the first example above, we have, on differentiating both members of $y' = -\frac{x}{y}$,

$$y'' = -\frac{y - xy'}{y^2}.$$

Substituting in this the value of y', we have

$$y'' = -\frac{y + \dfrac{x^2}{y}}{y^2} = -\frac{x^2 + y^2}{y^3}.$$

Making use of the original relation $x^2 + y^2 = 25$, we have finally

$$y'' = -\frac{25}{y^3}.$$

Differentiating this, we have for the third derivative $y''' = \frac{75\,y'}{y^4}$. Replacing y' by its value $-\frac{x}{y}$, we have $y''' = -\frac{75\,x}{y^5}$; and so on.

In the case of the second exercise above, we found $y' = \frac{5 - 2\,x - 2\,y}{2\,x + 2\,y}$. Before differentiating this, use should be made of the principle involved in Note 4, § 13, by writing it thus

$$y' = \frac{5}{2\,(x + y)} - 1.$$

Then $$y'' = \tfrac{5}{2}\,D_x(x + y)^{-1} = -\frac{5}{2}\,\frac{1 + y'}{(x + y)^2}.$$

Noting that $1 + y' = \frac{5}{2(x + y)}$, $y'' = -\frac{25}{4(x + y)^3}$.

It will be left as an exercise to show that

$$y''' = \frac{3 \cdot 5^3}{2^3(x + y)^5},\quad y^{\text{iv}} = -\frac{3 \cdot 5 \cdot 5^4}{2^4(x + y)^7},\ \cdots,$$
$$y^{(n)} = (-1)^{n+1}\frac{3 \cdot 5 \cdot 7 \cdots (2\,n - 3) \cdot 5^n}{2^n(x + y)^{2n-1}}.$$

An alternative mode of procedure is the following:

Starting with $x^2 + y^2 = 25$, we have, on differentiating,

$$x + yy' = 0. \tag{1}$$

$$\therefore\ y' = -\frac{x}{y}, \text{ as before.}$$

Differentiating (1) instead of the value of y', we have

$$1 + y'^2 + yy'' = 0. \tag{2}$$

$$\therefore\ y'' = -\frac{(1 + y'^2)}{y} = -\frac{1 + \frac{x^2}{y^2}}{y} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

To find y''', differentiate (2), obtaining

$$3\,y'y'' + yy''' = 0. \tag{3}$$

$$\therefore\ y''' = -\frac{3\,y'y''}{y} = -\frac{75}{y^5}.$$

The principal reason for this alternative method is that it avoids the necessity of differentiating a fraction, and this may prove simpler in practice, at times. The student should try it in the case of the other illustrative example

$$x^2 + 2\,xy + y^2 - 5\,x + 4 = 0.$$

EXERCISES

Find the first two derivatives of y with respect to x for

1. $x^3 + y^3 = a^3.$

2. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

4. $x^n + y^n = a^n.$

CHAPTER III

GEOMETRICAL AND OTHER APPLICATIONS

18. Maxima and minima. We have already noted (§ 7) that, when the derivative of a function is positive, the function increases with the variable; when it is negative, the function diminishes as the variable increases. If for a certain value, x_1, of the variable the derivative is zero, at that stage the function is neither increasing nor decreasing, as is also evident from the graph; for at the corresponding point of the graph the tangent is horizontal. A particularly interesting case arises when the derivative changes sign as the variable passes through the value x_1 for which the derivative vanishes. If, to fix the ideas, it was positive just before $x = x_1$ and negative just after (throughout this discussion we shall suppose x to be increasing algebraically), then $f(x_1)$ is larger than $f(x)$ for all neighboring values of x on either side of x_1. Thus consider

$$f(x) = \frac{x^4 - 8x^2}{8}.$$

Then $f'(x) = \dfrac{x^3 - 4x}{2} = \dfrac{x(x^2 - 4)}{2}$. Evidently $f'(x) = 0$ when $x = 0, 2, -2$. Moreover, from the form of $f'(x)$ in this case it is clear that, for small negative values of x, $f'(x)$ is positive; thus for $x = -\frac{1}{2}$, $f'(x) = \frac{15}{16}$. Similarly for small positive values of x, $f'(x)$ is negative; for $x = \frac{1}{2}$, $f'(x) = -\frac{15}{16}$. Here $f(-\frac{1}{2}) < f(0) > f(\frac{1}{2})$. Such a value of a function, which is greater than its neighboring values on either side, is said to be a *maximum* of the function. Thus in the special case under consideration, 0 is a maximum of $f(x)$ corresponding to the value $x_1 = 0$. In the graph we shall say that the point $O \equiv (0, 0)$ is a *maximum point* of the graph.

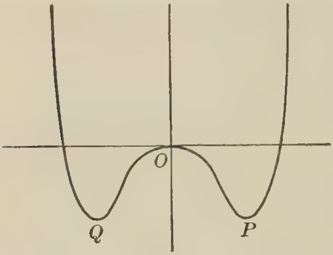

FIG. 6

If on the other hand $f'(x)$ is negative just before $x = x_1$ and positive just after, then $f(x_1)$ is smaller than $f(x)$ for all neighboring values of x on either side of x_1; and in this case $f(x_1)$ is said to be a *minimum* of $f(x)$. Thus, in the above example, we saw $f'(x) = 0$ when $x = 0, 2, -2$. From the form of $f'(x)$, the following facts are obvious:

For $0 < x < 2$, $f'(x)$ is negative; for example, when $x = 1$, $f'(x) = -\frac{3}{2}$.

For $x > 2$, $f'(x)$ is positive; for example, when $x = 3$, $f'(x) = \frac{15}{2}$.

Hence $f(2) = -2$ is a minimum of the function, and the point $P \equiv (2, -2)$ is a *minimum point* of the graph.

Similarly it is seen that $f(-2)$ is also a minimum of the function.

For $x < -2$, $f'(x)$ is negative; for example, when $x = -3$, $f'(x) = -\frac{15}{2}$.

For $-2 < x < 0$, $f'(x)$ is positive; for example, when $x = -1$, $f'(x) = \frac{3}{2}$. The point $Q \equiv (-2, -2)$ is also a minimum point of the graph.

If the graph of a function is known, the location of its maxima and minima is a matter of inspection. On the other hand, a knowledge of the maxima and minima of a function will assist materially in sketching the graph of the function.

REMARK. — Attention should be called to the fact that a maximum need not be the largest value of the function. In the case of the function considered above there is no largest value. For all values of x greater than 2 in absolute value, the function is positive. But still 0 is a maximum value, because $f(0)$ is greater than $f(x)$ for both positive and negative values of x near zero. Similarly a minimum need not be the smallest value of the function (although the minima in the case of the above illustration happen to be the smallest value of the function; this is a coincidence, not an essential).

EXERCISES

Find the maximum and minimum values of the following functions, and sketch their graphs:

1. x^2.
2. $3 - 2x^2$.
3. $2 + 3x - x^3$.
4. $\dfrac{8 + 2x^2 - x^4}{4}$.
5. $x^3 - 2x^2 + x + 2$.
6. $\dfrac{x}{1 + x^2}$.
7. $\dfrac{x^2}{1 + x^2}$.
8. $\dfrac{x^4 + 4x^2}{8}$.

19. Further facts concerning maxima and minima.

1. In looking for the values of the variable to which might correspond a maximum or a minimum of the function (we shall refer to these values as *critical* values of the variable), we used the roots of $f'(x) = 0$. It is perfectly possible for $f'(x_1)$ to equal zero without $f(x_1)$ being a maximum or minimum. *The essential thing is that $f'(x)$ should change sign as x passes through x_1.* This frequently happens when $f'(x_1) = 0$, but not necessarily. Thus if $f(x) = x^3$, then $f'(x) = 3x^2$. The latter vanishes for $x = 0$, but it is evidently positive for both positive and negative values of x. [Here $f(x)$ is always an increasing function; compare Fig. 7.] Hence $f(0)$ is neither a maximum nor a minimum. The graph of the function is said to have a *flex* or *point of inflection* at such a point 0. (See § 21.)

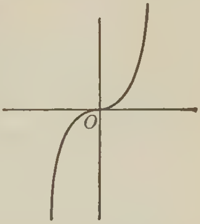

FIG. 7

2. Having found all the roots of $f'(x) = 0$, $f'(x)$ can be written as the product of its factors, thus

$$f'(x) = A(x - x_1)^k (x - x_2)^l \cdots.$$

Evidently if k is an odd number, $(x - x_1)^k$ will change sign as x grows from a value slightly smaller than x_1 to one slightly larger than it, since $x - x_1$ changes sign, while all the other factors do not change sign; hence $f(x_1)$ will be a maximum or minimum of $f(x)$. But if k is an even number, $(x - x_1)^k$ is positive always; hence $f'(x)$ will not change sign as x passes through x_1. In this case $f(x_1)$ is neither a maximum nor a minimum.

In the case of $$f(x) = \frac{x^4 - 8x^2}{8},$$

we saw $$f'(x) = \frac{x^3 - 4a}{2} = \frac{x(x-2)(x+2)}{2}.$$

Here each of the factors appears once (an odd number of times), hence each of the roots has a maximum or a minimum corresponding to it.

In the case of $f(x) = x^3$, $f'(x) = 3x^2$. Here we are dealing with a squared factor, and there is no corresponding maximum or minimum.

3. Once it is known that $f(x)$ has a maximum or minimum corresponding to the value x_1, its exact nature can be determined readily from the sign of $f''(x_1)$. Since $f''(x)$ is the derivative of $f'(x)$, $f'(x)$ is growing with x when $f''(x)$ is positive and it diminishes as x grows when $f''(x)$ is negative. But at a maximum and in its neighborhood the derivative of the function is diminishing as the variable increases; and at a minimum and in its neighborhood the derivative grows with the variable. Hence we have the very convenient criterion that

for a maximum $f'(x_1) = 0$ and $f''(x_1) < 0$;
for a minimum $f'(x_1) = 0$ and $f''(x_1) > 0$.

Thus, in the case of

$$f(x) = \frac{x^4 - 8x^2}{8}, \quad f'(x) = \frac{x^3 - 4x}{2}, \text{ and } f''(x) = \frac{3x^2 - 4}{2}.$$

Evidently

$$f'(0) = 0 \text{ and } f''(0) = -2; \text{ so } f(0) \text{ is a maximum.}$$
$$f'(2) = 0 \text{ and } f''(2) = 4; \quad \text{so } f(2) \text{ is a minimum.}$$
$$f'(-2) = 0 \text{ and } f''(-2) = 4; \text{ so } f(-2) \text{ is a minimum.}$$

This is consistent with what we found before.

If $f''(x_1) = 0$ at the same time that $f'(x_1) = 0$, one or the other methods of attack suggested before should be used.

In the case of $f(x) = x^3$, $f'(x) = 3x^2$, and $f''(x) = 3x$. Here $f'(0) = 0$ and $f''(0) = 0$. Our criterion gives us no information. But from the fact that $f'(x) = 3x^2$ is constant in sign it follows that $f(x)$ has no maximum or minimum value.

On the other hand, if $f(x) = x^4$, $f'(x) = 4x^3$, and $f''(x) = 12x^2$. Here again $f'(0) = 0$ and $f''(0) = 0$. So our criterion ceases to apply. But since x^3 changes sign with x and changes from $-$ to $+$ as x passes through 0, we know that $f(0)$ is a minimum for the function; as is also evident from the graph, Fig. 8.

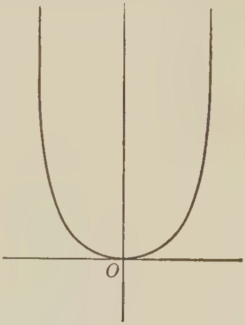

FIG. 8

4. Infinite values of a function, positive or negative, will not be considered as maxima or minima of the function, even if the derivative does change sign as the variable passes through a value for which the function becomes infinite. Thus if $f(x) = \frac{1}{x^2}$, $f'(x) = -\frac{2}{x^3}$. This is $+$ for $x < 0$, and $-$ for $x > 0$. But ∞ will not be considered a maximum of the function.

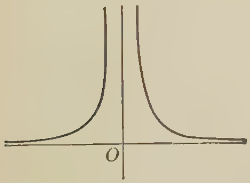

FIG. 9

5. If $f(x_1)$ is finite but $f'(x_1)$ is infinite and changes sign as x passes through x_1, $f(x_1)$ is a maximum if $f'(x)$ changes from $+$ to $-$, and a minimum if $f'(x)$ changes from $-$ to $+$. Thus if $f(x) = x^{\frac{2}{3}}$, $f'(x) = \frac{2}{3\,x^{\frac{1}{3}}}$. Here $f(0) = 0, f'(0) = \infty$. For negative values of x, $f'(x)$ is $-$; for positive values of x, $f'(y)$ is $-$. Hence $f(0)$ is a minimum of $f(x)$. This is also obvious from the graph.

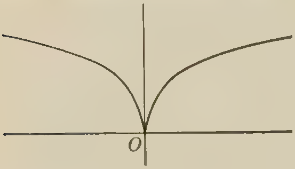

Fig. 10

6. If $f(x)$ is continuous, between any two consecutive maxima there is a minimum, and between any two consecutive minima there is a maximum. The proof of this will be left as an exercise.

Remark. — In what has gone before, the expressions "just before" and "just after," "slightly smaller than" and "slightly larger than," and the like have been used. The question naturally arises, Exactly what is meant by these? The answer is very simple. In the types of functions we are using here, $f'(x)$ changes sign only when x passes through a value for which $f'(x) = 0$ or ∞. If then we arrange the set of critical values of x (for which either of these cases arises) in order of algebraic size, say $x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n$, we shall mean by "just before x_i" or "slightly smaller than x_i" any value of x between x_{i-1} and x_i; by "just after x_i" or "slightly greater than x_i" any value of x between x_i and x_{i+1}. In the case of x_1, the smallest of the set, we shall mean by "just before" or "slightly smaller than" any value of x less algebraically than x_1, and by "just after" or "slightly great than x_n," the largest value of the set, any number larger than x_n. Thus, in the case of the function $f(x) \equiv \frac{x^4 - 8\,x^2}{8}$, the set of critical values was seen to be -2, 0, 2.

Any number algebraically smaller than -2 may be used as "just before -2."

Any number algebraically greater than -2 and less than 0 may be used as "just after -2" and as "just before 0"; and so on.

For practical purposes, this remark will be found very serviceable. For, excepting in rare instances, one will be able to use convenient integral or simple fractional values instead of confining oneself to values very near the critical values. On the other hand, one will be cautioned not to use numbers deviating too much from the critical values.

Similarly by a "sufficiently near" value to x_i we mean any value that lies between x_{i-1} and x_{i+1}, other than x_i itself. And by a "neighborhood of x_i" we mean any interval of values between x_{i-1} and x_{i+1} excluding x_i.

Summary. The several methods given above for finding the maxima and minima of a function are not mutually exclusive. In actual practice, the following mode of procedure is suggested:

I. Given $f(x)$, calculate $f'(x)$, and find the values of x for which $f'(x) = 0$ and $f'(x) = \infty$. Let these be

$$x_1,\ x_2,\ \cdots,\ x_i,\ \cdots,\ x_n$$

arranged in algebraic order.

II. Writing $f'(x)$ as the product of its factors,

$$f(x) \equiv A(x - x_1)^a (x - x_2)^b \cdots (x - x_i)^k \cdots (x - x_n)^l,$$

where A is a possible constant factor, and the exponents $a, b, \cdots, k, \cdots, l$ may be positive or negative,

$f(x_i)$ will surely not be a maximum or minimum if k is even,[1]

$f(x_i)$ will surely be a maximum or minimum if k is odd.

---

[1] If a factor appears with a fractional exponent, the factor will be said to appear an even or an odd number of times according as the numerator of the exponent is even or odd. Only cases in which the denominator of the fractional exponent is odd need be considered. For if it were even, we would be dealing with an even root of the corresponding factor $(x - x_i)$. Since this factor is positive on one side of x_i and negative on the other, the even root of it is imaginary on one side or the other. Hence there is no question of $f(x_i)$ being greater (or less) than $f(x)$ for neighboring values of x on both sides of x_i. See Ex. 4 of this set.

In the latter case we proceed to find $f''(x)$:

III. If $f''(x_i) < 0$, $f(x_i)$ is a maximum;
if $f''(x_i) > 0$, $f(x_i)$ is a minimum;
if $f''(x_i) = 0$ or ∞, no conclusion can be drawn.

In the latter case, and also when it is not practical to calculate $f''(x)$, we examine the behavior of $f'(x)$ in the neighborhood of x_i, when k is odd:

IV. Substituting values of x just less than x and just greater than x_i in $f'(x)$ (Remark),

if the resulting values are respectively $+$ and $-$, $f(x_i)$ is a maximum;

if the resulting values are respectively $-$ and $+$, $f(x_i)$ is a minimum.

EXERCISES

Find the maxima and minima of the following functions, and sketch their graphs:

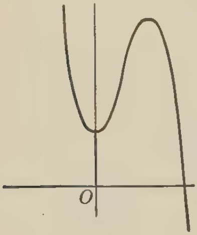

Fig. 11

1. $f(x) = 2 + 3x^2 - x^3$.

Here $f'(x) = 6x - 3x^2 = 3x(2 - x)$,
$f''(x) = 6 - 6x$.

The critical values, *i.e.* the roots of $f'(x) = 0$, are 0, 2. Since each of the corresponding factors x and $(2 - x)$ appears in $f'(x)$ to an odd degree, these values surely correspond to maxima or minima.

Since $f''(0) = +6$, $f(0) = 2$ is a minimum.
Since $f''(0) = -6$, $f(2) = 6$ is a maximum.

2. $f(x) = x^{\frac{2}{3}}(x^2 - 4)$.

To differentiate this, it is simpler to write $f(x)$ as follows:

$$f(x) = x^{\frac{8}{3}} - 4x^{\frac{2}{3}}.$$

$$\therefore f'(x) = \tfrac{8}{3}x^{\frac{5}{3}} - \frac{8}{3x^{\frac{1}{3}}} = \frac{8(x-1)(x+1)}{3x^{\frac{1}{3}}}.$$

$$f''(x) = \tfrac{40}{9}x^{\frac{2}{3}} + \frac{8}{9x^{\frac{4}{3}}} = \frac{8(5x^2+1)}{9x^{\frac{4}{3}}}.$$

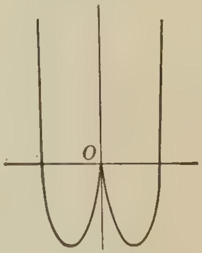

Fig. 12

The critical values are $-1, 0, 1$. Since each of the corresponding factors $(x+1)$, x, $(x-1)$ appears an odd number of times in $f'(x)$, these values surely correspond to maxima or minima.

Since $f''(-1) = +\frac{16}{3}$, $f(-1) = -3$ is a minimum.
Since $f''(0) = \infty$, another criterion must be used.
Since $f''(1) = +\frac{16}{3}$, $f(1) = -3$ is a minimum.

From fact 6, we should suspect that $f(0)$ is a maximum. That this is correct follows because $f'(-\frac{1}{2}) = 2\sqrt[3]{2}$ and $f'(\frac{1}{2}) = -2\sqrt[3]{2}$.

3. $f(x) = \dfrac{2x^3 + 3x^2 + 6x + 5}{5}$.

Here $f'(x) = \dfrac{6(x^2 + x + 1)}{5}$.

This has no real roots, and does not change its sign. Hence there are no maxima or minima of the function.

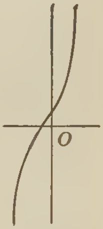

FIG. 13

4. Find the maximum and minimum points of the curve

$$25y^2 = x^2(4 - x^2)^3.$$

$$5y = \pm x(4 - x^2)^{\frac{3}{2}}.$$

$$5y' = \pm(4 - x^2)^{\frac{3}{2}} \mp 3x^2(4 - x^2)^{\frac{1}{2}} = \pm 4(1 - x^2)\sqrt{4 - x^2}.$$

The critical values are, arranged in order, $-2, -1, 1, 2$. Since the curve does not extend to the left of $x = -2$ or to the right of $x = 2$, the critical values -2 and 2 cannot lead to maxima or minima.

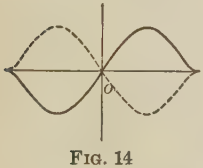

FIG. 14

It will be left as an exercise to show that

$$5y'' = \frac{12x(x^2 - 3)}{\pm\sqrt{4 - x^2}}.$$

For $x = -1$, y'' is $\dfrac{\pm 8\sqrt{3}}{5}$ according as the + or − sign is used in the value of y. Hence $x = -1$ corresponds to a minimum for that branch of the curve in which the + sign is used for y. (In the graph, this branch of the curve is represented by a continuous line.) While for the other branch (represented by the broken line in the graph) $x = -1$ corresponds to a maximum.

For $x = 1$, $y'' = \mp\dfrac{8\sqrt{3}}{5}$ according as the + or − sign is used in the value of y. Hence $x = 1$ corresponds to a maximum and a minimum respectively to the two branches.

The entire curve thus has the maximum points $\left(-1, \frac{3\sqrt{3}}{5}\right)$ and $\left(1, \frac{3\sqrt{3}}{5}\right)$, and the minimum points $\left(-1, -\frac{3\sqrt{3}}{5}\right)$ and $\left(1, -\frac{3\sqrt{3}}{5}\right)$.

5. $$f(x) = (x-1)^{\frac{2}{3}}(x+1)^3.$$

$$\therefore\ f'(x) = \frac{(x+1)^2(11x-8)}{3(x-1)^{\frac{1}{3}}}.$$

The critical values of x are $-1, \frac{8}{11}, 1$.

Since the factor $(x+1)$ occurs twice in $f'(x)$, -1 does not correspond to either a maximum or a minimum.

Each of the other factors appears an odd number of times in $f'(x)$; hence each has a maximum or minimum corresponding to it.

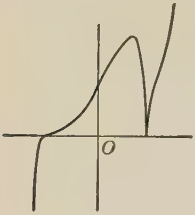

FIG. 15

The calculation of $f''(x)$ is not so simple here. So we shall use method contained in IV.

In the neighborhood of $x = \frac{8}{11}$, $\frac{(x+1)^2}{3(x-1)^{\frac{1}{3}}}$ is negative. For values of x slightly smaller than $\frac{8}{11}$, $11x - 8$ is negative, and for values of x slightly larger than $\frac{8}{11}$, $11x - 8$ is positive. Hence, as x passes through $\frac{8}{11}$, $f'(x)$ passes from + to −; that is, $x = \frac{8}{11}$ corresponds to a maximum.

In the neighborhood of $x = 1$, $(x+1)^2(11x-8)$ is positive. For values of x slightly smaller than 1, $(x-1)^{\frac{1}{3}}$ is negative, and for values of x slightly larger than 1, $(x-1)^{\frac{1}{3}}$ is positive. Hence, as x passes through 1, $f'(x)$ passes from − to +; that is, $x = 1$ corresponds to a minimum Compare the graph in Fig. 15.

6. $f(x) = x + \frac{1}{x} = \frac{x^2+1}{x}$.

7. $f(x) = x^2 + \frac{2}{x} = \frac{x^3+2}{x}$.

8. $f(x) = x^2 + \frac{1}{x^2} = \frac{x^4+1}{x^2}$.

9. $f(x) = \frac{4x^3 - x^4}{3}$.

10. $f(x) = \frac{x^5 - 15x^3}{54}$.

11. $f(x) = (x+1)^{\frac{1}{3}}(x-6)^2$.

12. $f(x) = x^{\frac{2}{3}}(x-5)$.

20. Problems involving maxima and minima. The following problems will illustrate:

Ex. 1. Separate 6 into two parts so that their product will be a maximum.

Let x be one part, then $6 - x$ will be the other, and their product will be $x(6 - x)$. To find its maximum we proceed as before:

$$y = x(6 - x) = 6x - x^2.$$
$$y' = 6 - 2x, \quad y'' = -2.$$

$y' = 0$ when $x = 3$. Since y'' is negative, we have a maximum. The two required parts are 3 and 3, and the maximum value is 9.

Ex. 2. The stiffness of a rectangular beam varies as its breadth and as the cube of its depth. What are the dimensions of the stiffest beam that can be cut from a log 10 inches in diameter?

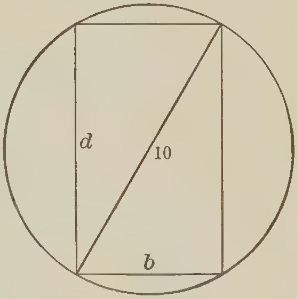

Fig. 16

Let $b \equiv$ breadth of beam.
$d \equiv$ depth of beam.
$s \equiv$ stiffness of beam.

Then $s = kbd^3$, where k is the constant factor of proportionality. Here the function to be maximized is given in terms of two variables b and d. But these are not independent of each other. They are connected by the relation

$$b^2 + d^2 = 100, \text{ or } b = \sqrt{100 - d^2}.$$

Hence we may write

$$s = kd^3\sqrt{100 - d^2}.$$

Then
$$s' = 3kd^2\sqrt{100 - d^2} - \frac{kd^4}{\sqrt{100 - d^2}}.$$

Equating this to zero, we have, on clearing of fractions,

$$3kd^2(100 - d^2) - kd^4 = 0, \text{ or } 300d^2 - 4d^4 = 0.$$
$$\therefore d = 0,\ 5\sqrt{3},\ -5\sqrt{3}.$$

Evidently $d = 0$ and $d = -5\sqrt{3}$ are of no interest here.

$d = 5\sqrt{3}$ is easily seen to correspond to a maximum. Along with this, we have $b = 5$; and the maximum stiffness is $k\,1875\sqrt{3}$.

These two problems are typical of those apt to arise. In the first, the function to be maximized or minimized is given directly in terms of a single variable. In the second, the function is given in terms of two variables, which, in turn, are connected by a relation, so that one of the variables can be expressed in terms of the other. The problem thus finally reduces to one of a single variable.

In more involved problems it may be desirable or necessary to introduce even a larger number of variables, say n, in order to obtain an expression for the function to be maximized or minimized. In the types of problems considered here, these n variables will be found subject to $(n - 1)$ relations, so that only one of them is actually independent.

REMARK 1. — Ex. 2 can be solved more simply by making use of the following general considerations:

(a) Since k is a positive constant, evidently kbd^3 will be a maximum or minimum whenever bd^3 is. So that we could have considered the function $s = bd^3$, without affecting the result. In general we may say that *any constant or non-vanishing factor in the function to be maximized or minimized may be omitted in the process.* If the omitted factor is positive, no further record of it need be kept. If it is negative, the final interpretation as to whether the function is at a maximum or minimum will have to take this factor into account.

(b) A further marked simplification may be affected by noting that since the radical sign in the expression for s is to be taken with the positive sign, s will be a maximum when s^2 is, and vice versa. But it is much simpler to differentiate the corresponding expression. Thus, making use of (a) and (b), we may write

$$\left(\frac{s}{k}\right)^2 \equiv f(d) = d^6(100 - d^2) = 100\,d^6 - d^8.$$

$$\therefore f'(d) = 600\,d^5 - 8\,d^7.$$

Equating to zero, we have at once $d = 0$ or $\pm\,5\sqrt{3}$.

REMARK 2. — An alternative method for solving problems like Ex. 2 (in which the function is given in terms of two variables subject to a known relation) may be given here. We had, writing $\frac{s}{k} \equiv S$,

$$S = bd^3 \quad \text{and} \quad b^2 + d^2 = 100.$$

Differentiating both relations with respect to d,

$$S' = d^3b' + 3\,bd^2 \quad \text{and} \quad bb' + d = 0.$$

From the second we have $b' = -\frac{d}{b}$. Putting this in $S' = 0$, we have, on clearing of fractions,

$$-d^4 + 3\,b^2d^2 = 0 \quad \text{or} \quad d^2 = 3\,b^2,$$

neglecting the useless value $d = 0$.

From $b^2 + d^2 = 100$, we have at once $b = 5$; whence $d = 5\sqrt{3}$.

This method is especially serviceable when we are interested in the relative values only of the two variables involved and not necessarily in their actual values.

Ex. 3. What are the relative dimensions of a cylindrical tin can of given content using the least amount of tin?

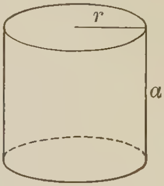

FIG. 17

If $r \equiv$ radius of base, $t \equiv$ amount of tin,
$a \equiv$ altitude, $V \equiv$ content,

we have

$$t = 2\,\pi ar + 2\,\pi r^2 \quad \text{and} \quad \pi ar^2 = V, \text{ a constant.}$$

Putting $a = \frac{V}{\pi r^2}$ in t, we have

$$t = \frac{2\,V}{r} + 2\,\pi r^2.$$

$$\therefore\ t' = -\frac{2\,V}{r^2} + 4\,\pi r.$$ Equating this to zero, we have

$$r^3 = \frac{V}{2\,\pi} \quad \text{or} \quad r = \sqrt[3]{\frac{V}{2\,\pi}}.^{1}$$

$$\therefore\ a = \frac{V}{\pi r^2} = \sqrt[3]{\frac{V^3}{\pi^3} \cdot \frac{4\,\pi^2}{V^2}} = \sqrt[3]{\frac{4\,V}{\pi}} = 2\sqrt[3]{\frac{V}{2\,\pi}}.$$

That is, $a = 2\,r$.

---

[1] That this value of r renders t a minimum is more or less obvious from the nature of the problem. If desired, the student will find little difficulty in passing to t'', and noting that this is positive for this critical value of r.

While the work involved here is not complicated, the simplification that could be effected by using Remark 2 is quite worth while. Thus we have

$$T \equiv \frac{t}{2\pi} = ar + r^2 \quad \text{and} \quad ar^2 = \frac{V}{\pi} = \text{constant}.$$

Differentiating with respect to r,

$$T' = ra' + a + 2r \quad \text{and} \quad r^2a' + 2ar = 0.$$

From the second we have $a' = -\frac{2a}{r}$; putting this in the first and equating to zero,

$$-2a + a + 2r = 0, \text{ or } a = 2r.$$

Ex. 4. A man in a boat on a lake wishes to reach an inland point in the shortest time, by rowing to some point on the shore, and walking the rest of the way. At what point must he land, if he can row v_1 miles per hour, and walk v_2 miles per hour on land? The shore is supposed to be straight.

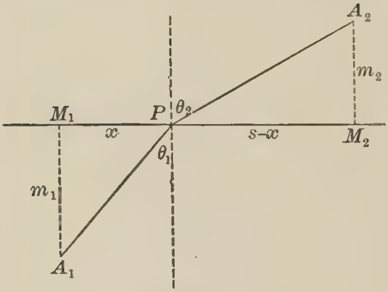

FIG. 18

If A_1 is the original position of the man and A_2 is his goal, let $m_1 \equiv A_1M_1$, the distance from A_1 to the shore, $m_2 \equiv A_2M_2$, the distance from A_2 to the shore. If P is his landing place, and if $M_1M_2 \equiv s$, let $x \equiv M_1P$; then $s - x = PM_2$.

$\theta_1 \equiv$ angle which A_1P makes with the normal to the shore.

$\theta_2 \equiv$ angle which PA_2 makes with the normal to the shore.

The time consumed in rowing is $\frac{\sqrt{x^2 + m_1^2}}{v_1}$.

The time consumed in walking is $\frac{\sqrt{(s-x)^2 + m_2^2}}{v_2}$.

The total time consumed is

$$T=\frac{\sqrt{x^2+m_1^2}}{v_1}+\frac{\sqrt{(s-x)^2+m_2^2}}{v_2}.$$

To render this a minimum, differentiate with respect to x.

$$T'=\frac{x}{v_1\sqrt{x^2+m_1^2}}-\frac{s-x}{v_2\sqrt{(s-x)^2+m_2^2}}=\frac{\sin\theta_1}{v_1}-\frac{\sin\theta_2}{v_2}.$$

This vanishes when

$$\frac{\sin\theta_1}{v_1}=\frac{\sin\theta_2}{v_2}. \tag{1}$$

That this value of x or this relation between θ_1 and θ_2 renders T a minimum is evident from the nature of the problem. But it is not difficult to show that, since the radicals used here must be taken with the positive sign, T'' is positive, as it should be for a minimum. This will be left as an exercise.

The result (1) is interesting, because the path traced by the man to pass from A_1 to A_2 in the least time is entirely analogous to that of a ray of light in going from a point A_1 in one medium, in which its velocity is v_1, to a point A_2 in a second medium, in which its velocity is v_2.

EXERCISES

1. What number exceeds its square by the greatest amount?

2. Divide 5 (also n) into two parts such that the sum of their squares is a minimum.

3. Divide 5 (also n) into two parts such that the sum of the square of twice one part and the square of the other is a minimum.

4. A gutter of maximum capacity is to have its bottom and sides 2 inches each, the sides to make equal angles with the bottom as indicated in Fig. 19. What should be the distance across the top?

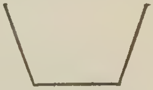

FIG. 19

5. If the strength of a rectangular beam is proportional to its breadth and to the square of its depth, find the dimensions of the beam of maximum strength that can be cut from a log which is an elliptic cylinder whose cross section is the ellipse $\frac{2x^2}{27}+\frac{y^2}{12}=1$.

6. Find the relative dimensions of a conical tent of given volume that requires the least material.

7. A sector is cut from a circle and bent so as to form the lateral surface of a cone. What must be the angle of the sector so that the cone may have a maximum content?

8. Show that an equilateral triangle has the maximum area of all the isosceles triangles inscribed in a given circle.

9. Show that a square has the maximum area (also perimeter) of all the rectangles inscribed in a given circle.

10. Find the dimensions and area of the rectangle of maximum area that can be inscribed in the ellipse $b^2x^2+a^2y^2=a^2b^2$.

11. Find the rectangle of maximum perimeter inscribed in the ellipse $b^2x^2+a^2y^2=a^2b^2$.

12. Find the relative dimensions of the cylinder of maximum volume inscribed in a sphere of radius a. What is its volume?

13. Find the relative dimensions of the right circular cone of maximum volume inscribed in a sphere of radius a. What is its volume?

14. Find the relative dimensions of the right circular cone of minimum volume that can be circumscribed about a sphere of radius a. What is its volume?

15. Equal squares are cut from the corners of a rectangular piece of cardboard 8 by 3 inches. The projections are then

turned up so as to form an open box. What must be the size of the squares cut out so that the content of the open box may be a maximum?

16. What are the relative dimensions of a cylindrical tin can using a given amount of tin that shall have a maximum content? Compare illustration Ex. 3.

17. What are the relative dimensions of the most economical cylindrical tin cup?

18. What are the relative dimensions of a rectangular parallelopiped with a square base so that for a given content the total area is a minimum?

19. What are the relative dimensions in the parallelopiped of Ex. 18 if the area of one of the bases is not included in the total area (for example, in the case of an open square box)?

20. The size of a package is restricted so that the sum of the length and girth must not exceed 72 inches. What shaped package will have the maximum content?

Hint. — The shapes to try are a sphere, a rectangular parallelopiped with a square base, and a right circular cylinder. Find the most advantageous dimensions in each of these cases and their corresponding contents.

21. A man in a launch on a lake, 3 miles from a straight shore, wishes to reach a point 10 miles along the shore. If he can travel 20 miles an hour in the launch, and an auto which follows him along the shore ready to pick him up at any time can make 29 miles an hour, at what point should he land so as to reach his destination in the least time?

22. In Ex. 21 consider the general case where the man starts from a point m_1 miles from the shore, the objective point is s miles from the point on the shore nearest the starting point, the launch travels with the speed v_1, and the automobile with the speed v_2. What rôle does the value of s play?

23. A boy at A_1 distant m_1 feet from M_1, the nearest point of a straight fence, wishes to reach in the shortest time a second point A_2 on the same side and distant m_2 feet from M_2, the nearest point of the fence from it. He can run v ft./sec. and must touch the fence once. Show that the path of the boy is that of a ray of light reflected by a plane surface. Compare illustrative Ex. 4.

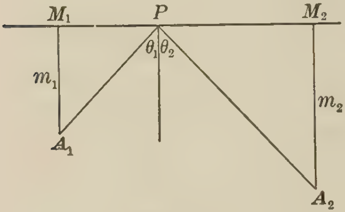

FIG. 20

24. A conical glass is filled with water, and a marble is carefully dropped into it. If the radius of the base of the cone is 3 inches and its altitude 4 inches, what must be the radius of the marble so as to displace the greatest amount of water?

HINT. — Let the distance from the center of the spherical marble to the vertex of the cone be h. The volume of the water displaced is that of the segment of the sphere whose altitude is $r-(h-4)$.

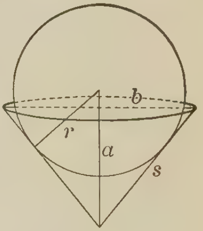

FIG. 21

25. In Ex. 24, consider the general case where the radius of the base of the cone is b, its altitude is a, and its slant height s.

26. The distance between two spheres of radii r_1 and r_2, respectively, is s. At what point on the line of centers of the two spheres must a light be placed so as to illuminate the greatest amount of surface on the two spheres?

27. Show that if P is a variable point on the curve $y=f(x)$ and A is a point not on the curve, the line AP must be normal to the curve at P if AP is to be either a maximum or a minimum.

28. A weight W is to be raised by means of a lever weighing w pounds per linear foot. The weight is suspended at a point a feet from the end of the lever, which is the fulcrum or point of support. What should be the length of the lever so that the force required to raise the weight if applied at the free end of the lever is to be a minimum?

29. The formula for the efficiency of a screw is

$$E = \frac{h(1 - \mu h)}{h + \mu},$$

where μ is the coefficient of friction, a constant, and h is the pitch of the screw. For what value of h is the efficiency a maximum?

30. The velocity of waves of length λ in deep water is

$$k\sqrt{\frac{\lambda}{a} + \frac{a}{\lambda}}$$

where a and k are constants. For what value of λ is the velocity a minimum?

31. The work done by a voltaic cell of constant electromotive force E and constant internal resistance r in sending a steady current through an external circuit with resistance R is

$$W = \frac{kE^2R}{(r + R)^2}$$

in a given time. Here k is a constant depending upon the units selected. For what value of R is W a maximum?

32. When a battery of mn cells is joined up in m rows, of n cells each connected in series, and the rows are joined in parallel, the current is equal to

$$i = \frac{mnE}{mR + nr},$$

where E, the electromotive force of each cell, r, the internal resistance of each cell, and R, the external resistance are constants. Show that the current is a maximum when $R = \frac{nr}{m}$, that is, when the internal resistance is equal to the external resistance.

21. Concavity; point of inflection (or flex). We have already noted (§ 19, 3) that when $f''(x)$ is positive, $f'(x)$ increases with x. In the graph of $f(x)$ this means that, as we move along the curve from left to right, the tangent revolves in counterclockwise sense; that is, the curve is *concave upward.* Again, if $f''(x)$ is negative, $f'(x)$ decreases as x increases; for the graph, the tangent revolves in clockwise sense. The curve is *concave downward.* This is consistent with what we saw before (§ 19, 3), $f''(x)$ is positive at and in the neighborhood of a minimum, and negative in the case of a maximum point.

If, as we move along the graph of a function, the sense of the concavity changes from upward to downward, or vice versa, the point on the curve at which the change takes place is a *point of inflection* or, more simply, a *flex* of the curve.[1] As the variable passes through the value corresponding to a flex, $f''(x)$ changes signs. This can occur only (in the case of the functions we shall meet) by having $f''(x) = 0$ or $f''(x) = \infty$. Thus, if

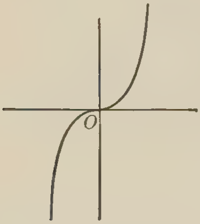

FIG. 22

$$f(x) = x^3, f'(x) = 3\,x^2 \text{ and } f''(x) = 6\,x.$$

$$\begin{aligned} \text{Here } f''(x) &= 0 \text{ when } x = 0,\\ f''(x) &< 0 \text{ when } x < 0,\\ f''(x) &> 0 \text{ when } x > 0. \end{aligned}$$

The origin is a flex for the graph. For points to the left the concavity is downward, for those to the right it is upward.

Again consider $f(x) = x^{\frac{1}{3}}$.

$$f'(x) = \frac{1}{3\,x^{\frac{2}{3}}},\ f''(x) = -\frac{2}{9\,x^{\frac{5}{3}}}.$$

The origin is also a flex for the graph, since $f''(x)$ is positive for

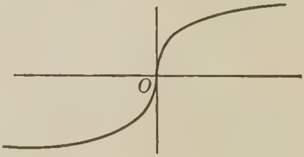

FIG. 23

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[1] At a point of inflection the curve crosses its tangent.

points to the left of the origin, and negative for points to the right of it. At the origin, $f''(x) = \infty$.

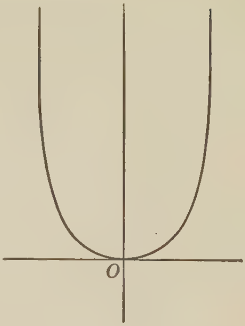

FIG. 24

Moreover $f''(x_0) = 0$ or $f''(x_0) = \infty$ is not sufficient to cause the graph to have a flex at the point $(x_0, f(x_0))$. Thus, if $f(x) = x^4$, $f''(x) = 12x^2$. This vanishes for $x = 0$; but it is positive for all other values of x. So there is no flex at the origin. Again if $f(x) = x^{\frac{2}{3}}$, $f''(x) = -\frac{2}{9x^{\frac{4}{3}}}$. Here $f''(0) = \infty$; but it is negative for all other values of x. Hence there is no flex at the origin.

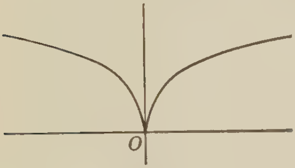

FIG. 25

Consider now

$$f(x) = x^{\frac{1}{3}}(x+4) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}},$$

$$f'(x) = \tfrac{4}{3}x^{\frac{1}{3}} + \tfrac{4}{3}x^{-\frac{2}{3}} = \frac{4(x+1)}{3x^{\frac{2}{3}}},$$

$$f''(x) = \tfrac{4}{9}x^{-\frac{2}{3}} - \tfrac{8}{9}x^{-\frac{5}{3}} = \frac{4(x-2)}{9x^{\frac{5}{3}}}.$$

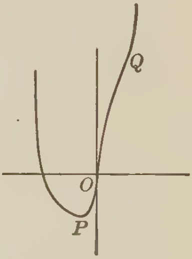

FIG. 26

Here $f''(x) \lesseqgtr 0$ when $x \lesseqgtr 2$. Hence $(2, 6\sqrt[3]{2})$ is a flex.

$f''(x) = \infty$ when $x = 0$. It is positive just before and negative just after. So the origin also is a flex.

There is a minimum corresponding to $x = -1$.

REMARK. — The problem of finding the flexes of the graph of a function is analogous to that of finding its maximum and minimum points. In the former case we are concerned with the changes of sign of $f''(x)$ just as in the latter we are interested in the behavior of $f'(x)$. The whole discussion of §§ 18 and 19 could be repeated here. But the modifications to be made in it are so obvious it does not seem necessary.

After finishing the exercises of this section, the student is urged to take up the general problem of sketching curves, which is treated in the Appendix.

EXERCISES

Find the maximum and minimum points, and the flexes of the following curves. Sketch the curves.

1. $y = x^3 - 3x + 2.$
2. $10y = x^3(x - 5)^2.$
3. $y = 3x^4 + 5x^3 - 8.$
4. $y = 3x^4 - 4x^3.$
5. $y = \dfrac{6}{3 + x^2}.$
6. $y = \dfrac{108}{80 + 3x^4}.$
7. $y = \dfrac{x}{1 + x^2}.$
8. $y = \dfrac{x^2}{1 + x^2}.$
9. $y = \dfrac{x^3}{3 + x^2}.$
10. $y = \dfrac{x^3}{x + 2}.$
11. $y^2 = x^2(4 - x^2).$
12. $y^2 = 4(1 - 2x^2)^3.$
13. $y^2 = (1 - x^{\frac{2}{3}})^3.$
14. $y = x - 2\sqrt{x}.$

15. Show that for $y = x^n$, where n is a positive integer, the origin is a minimum or a flex according as n is even or odd.

16. Show that for $y = x^{\frac{p}{q}}$, where $\dfrac{p}{q}$ is a positive fraction reduced to its lowest terms and q is odd, the origin is a minimum or a flex according as p is even or odd.

22. Tangent; subtangent; normal; subnormal. Let P be a point (x_0, y_0) on the curve $y = f(x)$; and PK and PR the tangent and normal at P respectively. We shall be interested in finding the following lengths:

1. $PK \equiv T_x$, the part of the tangent extending from P to the x-axis.

As noted in § 6 the slope of the tangent at P is

$$\tan\theta = f'(x_0) = y'_0.$$

$$\therefore \sec\theta = \sqrt{1 + y_0'^2},$$

$$\csc\theta = \frac{\sqrt{1 + y_0'^2}}{y'_0}.$$

In the right triangle PKQ, $PK = PQ \csc\theta$. Hence

$$T_x = \frac{y_0}{y'_0}\sqrt{1 + y_0'^2}. \quad (1)$$

2. $PV \equiv T_y$, the part of the tangent extending from P to the y-axis.

In the right triangle PVW, $PV = VW \sec\theta$. Hence

$$T_y = x_0\sqrt{1 + y_0'^2}. \quad (2)$$

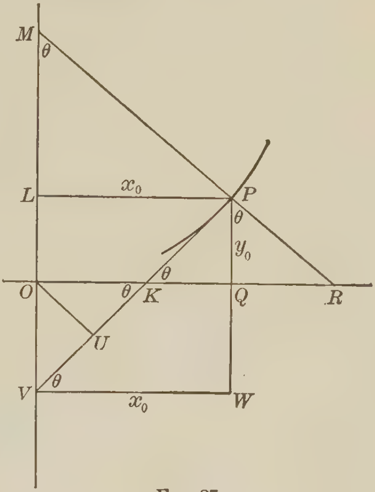

FIG. 27

3. $KV \equiv T.$, the part of the tangent included between the coördinate axes.

$$T. = T_y - T_x = \frac{x_0 y'_0 - y_0}{y'_0}\sqrt{1 + y_0'^2}. \quad (3)$$

4. $KQ \equiv S.T.$, the *subtangent*, which is the projection of T_x upon the x-axis.

In the right triangle PKQ, $KQ = PQ \cot\theta$. Hence

$$S.T. = \frac{y_0}{y'_0}. \quad (4)$$

5. $PR \equiv N_x$, the part of the normal extending from P to the x-axis.

Using the right triangle PQR, the student should show that

$$N_x = y_0\sqrt{1 + y_0'^2}. \quad (5)$$

6. $PM \equiv N_y$, the part of the normal extending from P to the y-axis.

Using the right triangle MLP,

$$N_y = \frac{x_0}{y_0'}\sqrt{1 + y_0'^2}. \tag{6}$$

7. $MR \equiv N.$, the part of the normal included between the coördinate axes.

$$N. = N_x + N_y = \frac{x_0 + x_0 y_0'}{y_0'}\sqrt{1 + y_0'^2}. \tag{7}$$

8. $QR \equiv S.N.$, the *subnormal*, which is the projection of N_x upon the x-axis.

Using the right triangle PQR,

$$S.N. = y_0 y_0'. \tag{8}$$

9. $OU \equiv p_t$, the distance of the tangent from the origin.

In the right triangle OUK, $OU = OK \sin\theta$.

$$OK = OQ - KQ = x_0 - \frac{y_0}{y_0'};\ \sin\theta = \frac{y_0'}{\sqrt{1 + y_0'^2}}.\quad \text{Hence}$$

$$p_t = \frac{x_0 y_0' - y_0}{\sqrt{1 + y_0'^2}}. \tag{9}$$

REMARK. — In the formulas of this section, we are interested, at this time, in the absolute values of various lengths, and not in their algebraic signs. It may be added that these formulas are given here rather for reference than that they be memorized. They are obtained so readily from Fig. 27, it is suggested the student make the effort to remember how to deduce them instead of memorizing the formulas themselves.

EXERCISES

1. Show that in the case of the parabola $y^2 = 4x$, the subtangent is twice the abscissa of the point of contact, and the subnormal is the same for all points of the curve.

2. Show that the part of the tangent to the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ included between the axes is a constant for all points of the curve.

3. Show that the part of any tangent to the equilateral hyperbola $xy = k^2$, included between the coördinate axes, is bisected by the point of contact.

4. Show the sum of the intercepts of any tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ on the coördinate axes is equal to a, a constant.

5. Find the lengths of the tangent, normal, subtangent, subnormal, and perpendicular from the origin to the tangent for

(a) $y = \dfrac{1}{1 - x}$ at the point whose $x = 2$.

(b) $xy^2 = 8$ at the points whose $x = 2$.

(c) $y^2 = \dfrac{x^3}{4 - x}$ at the points whose $x = 2$.

6. Find the tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, whose parts included between the coördinate axes are bisected by the point of contact.

7. Find the tangents to $y = 3\,x^3 - 4\,x^2$ which cut the axis of x at an angle of 45°.

8. Find the normals to $y^2 = 4\,x$ whose intercepts on the x-axis are equal to 6.

9. Show that the subtangent is equal to n times the abscissa of the point of contact, in the case of the curve $y^n = ax$. Using this fact, deduce a simple method for constructing the tangent to the curve at a given point. Illustrate in the special cases where $n = 2, 3, -1, \frac{1}{2}, \frac{2}{3}$.

23. Parametric equations. At times the equation of a curve may not be given in the form $f(x, y) = 0$ or $y = \phi(x)$, but x and y may each be given as a function of a third variable; the latter, in such cases, is called a *parameter*. Thus the equations

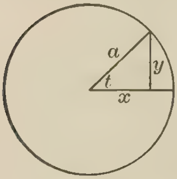

FIG. 28

$$x = a \cos t, \quad y = a \sin t \tag{1}$$

evidently represent the circle $x^2 + y^2 = a^2$, as may be seen by squaring both equations and adding.

In an analogous way the student may show that the equations

$$x = a \cos t, \quad y = b \sin t \tag{2}$$

represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the equations

$$x = a \sec t, \quad y = b \tan t \tag{3}$$

represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b} = 1.$

In general, to obtain the equation of a curve when the coördinates of its points are given in terms of a parameter, the parameter must be eliminated between the equations determining the coördinates.

As a further exercise, the student may show that the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is determined by the parametric equations

$$x = a \cos^3 t, \quad y = a \sin^3 t. \tag{4}$$

At times, however, it is more convenient to study the curve by using its parametric equations only. In such cases we should like to know $D_x y$ as a function of the parameter. To do this we proceed thus:

From $\qquad x = f(t), \quad y = \phi(t),$

we have $\qquad D_t x = f'(t), \quad D_t y = \phi'(t).$

But considering t as a function of x, determined by the first of the parametric equations, we have from formula VI (§ 11)

$$D_xy = D_ty D_xt.$$

But $D_xt = \dfrac{1}{D_tx}$ (formula VII, § 11). Hence

$$\textbf{X.}\qquad D_xy = \frac{D_ty}{D_tx} = \frac{\phi(t)}{f'(t)} = \frac{y'}{x'}.$$

Here the primes indicate derivatives with respect to the parameter.

For the second derivative D_x^2y we note that

$$D_x^2y = D_x(D_xy) = D_t(D_xy)D_xt = \frac{D_t(D_xy)}{D_tx}.$$

Using X, and applying the rule for the differentiation of a fraction, we have

$$\textbf{XI.}\qquad D_x^2y = \frac{D_t(D_xy)}{D_tx} = \frac{x'y'' - x''y'}{x'^3}.$$

The parametric equations of the curves mentioned so far in this section involve trigonometric functions of the parameter. We shall return to the consideration of these curves when we have learned to differentiate such functions. Other curves, whose equations in parametric form are found serviceable in practice, will be introduced from time to time. To illustrate the utility of having the equation of a curve in parametric form, we shall discuss the *folium of Descartes*. Its equation in rectangular coördinates is

$$x^3 + y^3 = 3\,axy.$$

Putting $y = tx$, we have

$$(1 + t^3)x^3 - 3\,atx^2 = 0$$

$$\therefore x = 0,\ 0,\ \frac{3\,at}{1 + t^3}.$$

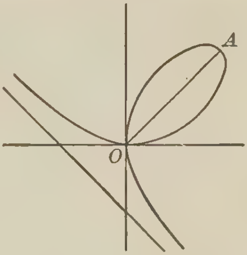

FIG. 29

The fact that two of the roots are 0 means that any line through the origin, $y = tx$, meets the curve in two coincident points at the origin; that is, the origin is a double point for the curve. The third value of x is 0 for $t = 0$ and $t = \infty$; that is, the axes of x and y are tangent to the curve at the origin.

Taking $x = \frac{3\,at}{1+t^3}$ as the value of x corresponding to any value t, we have for the corresponding value of y, $y = \frac{3\,at^2}{1+t^3}$. Hence the parametric equations of the folium of Descartes are

$$x = \frac{3\,at}{1+t^3}, \quad y = \frac{3\,at^2}{1+t^3}, \tag{5}$$

where the parameter t represents the slope of the radius vector to the corresponding point on the curve.

For $t = -1$, $x = \infty$ and $y = \infty$; that is, the curve goes to infinity in a direction that has the slope -1. Hence to find the equation of the asymptote to the curve we find for what value of b the line $y = -x + b$ cuts the curve in less than two finite points. See III in Appendix. Combining this with $x^3 + y^3 = 3\,ax$, we have

$$3(a+b)x^2 - 3\,b(a+b)x + b^3 = 0.$$

If $a + b = 0$, or $b = -a$, both roots of this equation are ∞.

Hence $$y = -x - a \text{ or } x + y + a = 0$$

is an asymptote to the curve.

As an exercise the student should show that OA, the greatest length of the loop, is $\frac{3\,a}{\sqrt{2}}$.

CHAPTER IV

PHYSICAL APPLICATIONS

24. Velocity; rates. As already noted in § 8, by the velocity of a particle moving in a straight line is meant the time derivative of the variable representing the position of the particle at any instant; thus, if the position of the particle is given by $s = f(t)$, where t is the time, the velocity at any instant t is $v = f'(t)$.

By analogy the time derivative of any variable may be considered the *rate of change* of that variable.

Since $D_xy = \dfrac{D_ty}{D_tx}$ (§ 23), we may say that D_xy, the derivative of y with respect to x, represents the *relative rate of change* of y to that of x.

We thus see that the "quantitative" property asked for in § 3 is given precisely by the derivative. Hence the relative rate of change at any instant, asked for there, is equal to the slope of the tangent to the graph at the corresponding point.

When the relation connecting two or more variables is determined by an equation, the result of differentiating both members of this equation with respect to the time is a new equation giving a relation between the derivatives of the variables with respect to the time (*i.e.* their rates of change) and the variables themselves. Then the rate of change of any one of the variables at any stage can be determined from this relation when the rates of change of the others are known. By "at any stage" we mean any instant when the variables have a given set of values. In the following exercises, the mode of procedure is to set up a relation between the variable, whose rate of change at a given stage is desired, and one or more variables whose rates of change are known. This is then differentiated with respect to the time and the required derivative is solved for.

Ex. 1. A man 6 feet tall starts to walk away from a pole at the rate of 3 miles per hour. If there is a light attached to the pole 18 feet from the ground, how rapidly is his shadow growing at any time?

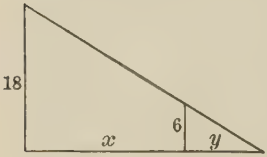

FIG. 30

If x represents the distance gone by the man in the time t, and y the length of the shadow at that instant, then by similar triangles, $\frac{x+y}{y}=\frac{18}{6}$ or $y=\frac{x}{2}$. This is the required relation. Denoting differentiation with respect to the time by a prime, we have

$$y'=\frac{x'}{2}.$$

But $x' = 3$ mi./hr. $\therefore y' = \frac{3}{2}$ mi./hr.

Here it is to be noted that the shadow grows uniformly; that is, y' is a constant, and does not depend upon x or y or t.

Ex. 2. If, in Ex. 1, the man starts 10 feet from the foot of the pole, and walks at the same rate as before but in a direction at right angles to the line joining his initial position and the pole, how rapidly is the shadow growing 1 minute after starting? Also 2 minutes after starting? After he has walked 25 feet?

Here, in the similar triangles of Fig. 30, x is to be replaced by $\sqrt{x^2+100}$.

Ex. 3. A man is walking along a horizontal plane away from the base of a monument 48 feet high. If he walks at the rate of 3 mi./hr., at what rate is he going from the top of the monument after having walked 36 feet? 64 feet? one minute?

Ex. 4. Water is flowing into a cone 10 ft. deep and 40 ft. across the top at the rate of 36 cu. ft./min. How fast is the surface of the water rising when it is 5 ft. above the apex of the cone? At the end of one minute?

Here $V = \frac{1}{3}\pi r^2 h = \frac{4}{3}\pi h^3$, since $r = 2h$. $\therefore\ V' = 4\pi h^2 h'$.

Ex. 5. If water is flowing into a cylindrical tank of diameter 20 ft. at the rate of 36 cu. ft./min., show that the rise of the surface of the water in the tank is uniform, and find its rate.

Ex. 6. The lower end of a ladder 26 ft. long is being drawn away from a wall at the rate of 3 ft./sec. How fast is the upper end, resting against a vertical wall, descending, when the lower end is 10 ft. from the wall?

Ex. 7. The diameter of a spherical balloon is growing at the uniform rate of 2 in./sec. At what rate is its volume increasing when the diameter is 1 ft.?

Ex. 8. A stone is dropped into a calm lake, sending out a series of concentric circular ripples. If the diameter of the outer ripple is growing uniformly at the rate of 5 ft./sec., how rapidly is the disturbed area growing at the end of 3 seconds?

Ex. 9. The legs of a right triangle are growing at the rate of 1 in./sec. and 2 in./sec. respectively. At what rate is the hypotenuse growing when the legs are 3 in. and 4 in. respectively?

Ex. 10. If one of the legs of a right triangle grows at the rate of 1 in./sec. and the other diminishes at the rate of 2 in./sec., at what rate is the hypotenuse changing when the legs are 5 in. and 12 in. respectively? 8 in. and 6 in. respectively?

Ex. 11. A train is moving along an elevated track 20 ft. high at the rate of 20 mi./hr. Immediately below a truck is moving along the street at 8 mi./hr. How fast are the train and truck separating 1 minute after the former was immediately above the latter? 3 minutes after?

Answer these questions in case the truck is moving along a cross street at right angles to the elevated track.

Ex. 12. The cylinder of a steam engine is 18 inches in diameter. What is the speed of the piston when steam is entering the cylinder at the rate of 27 cu. ft./sec.?

Ex. 13. Sand is allowed to heap up in the shape of a conical mound in which the altitude is $\frac{1}{3}$ the diameter of the base. If the mound is growing at a constant rate, show that the rate of growth of its height is inversely proportional to the square of the height. If the constant rate of growth of the mound is 18 cu. ft./min., at what rate is the height increasing when the latter is 4 ft.?

Ex. 14. The altitude of a cylinder is increasing at the rate of 2 in./sec., while the radius of its base is decreasing 1 in./sec. Is the volume increasing or decreasing, and at what rate, at the instant when the altitude is 4 in. and the radius of the base is 13 in.? What is its rate 3 seconds later? 5 seconds later? Does the volume attain a maximum or minimum value; if so, when?

25. Velocity; curvilinear motion. In the previous section and in § 8, velocity of motion in a straight line was considered.

If the motion is not in a straight line, the direction of the motion at any instant as well as its intensity enters into the expression for the velocity. If, as in the previous case, the distance s covered by the moving particle is considered as a function of the time, as it may always be, the velocity at any instant is still D_ts, but measured in the direction of motion of the particle along the tangent to the path at the point occupied by the particle at that instant.

Thus, if the particle is moving along the curve QR in the direction of the arrow, its velocity at the instant when it is at P is the value of D_ts at that instant, and measured along the tangent at P in the direction indicated. A quantity, which has both a numerical value and a direction, is known as a *vector quantity*. Velocity is a vector quantity. A distinction is made between *speed* and *velocity*, the former being the term used for the numerical value of the latter; it is the absolute value of D_ts. Velocity is determined by the two elements speed and direction.

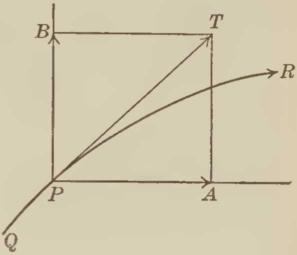

FIG. 31

In rectilinear motion the sign of D_ts determines the direction of the velocity along the line of motion.

In curvilinear motion two equations are necessary to determine the velocity of the moving particle. In practice it is found most convenient to express the velocity of the particle at any instant in terms of its projections on the coördinate axes, PA and PB in the figure. Calling these projections v_x and v_y respectively, it follows that

$$\left.\begin{aligned} &\textit{speed} \equiv |v| = +\sqrt{v_x^2 + v_y^2}.\\ &\textit{inclination of velocity} \equiv \theta = \cos^{-1}\frac{v_x}{|v|} = \sin^{-1}\frac{v_y}{|v|} = \tan^{-1}\frac{v_y}{v_x}\end{aligned}\right\} \quad (1)$$

If the motion of the particle is given in the form

$$x = f(t), \quad y = \phi(t),$$

then

$$\left.\begin{aligned} &v_x = D_tx = f'(t) = x', \quad v_y = D_ty = \phi'(t) = y'.\\ &|v| = \sqrt{[f'(t)]^2 + [\phi'(t)]^2} = \sqrt{x'^2 + y'^2}.\end{aligned}\right\} \quad (2)$$

Ex. 1. A projectile is hurled with a speed of v_0 ft./sec. in a direction making an angle α with the horizontal. Find its path and velocity.

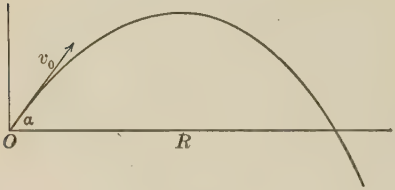

FIG. 32

The equations of motion of the projectile are (as we shall deduce later, § 40), if the starting point is taken as the origin,

$$\left.\begin{aligned} x &= v_0 t \cos\alpha, \\ y &= -16t^2 + v_0 t \sin\alpha. \end{aligned}\right\} \quad (3)$$

$$\therefore\ v_x = v_0 \cos\alpha,$$
$$v_y = -32t + v_0 \sin\alpha.$$

The rectangular equation of the path is obtained by eliminating t between equations (3), its parametric equations. Doing this, we find

$$y = -\frac{16}{v_0^2\cos^2\alpha}x^2 + \tan\alpha\cdot x, \quad (4)$$

which is a parabola through the origin with its axis vertical and its vertex at its highest point.

Its speed $\equiv |v| = \sqrt{v_0^2 + 1024t^2 - 64v_0 t \sin\alpha}$.

Show that this never vanishes if $\alpha \neq \frac{\pi}{2}$, and has a minimum at the same instant that y is a maximum.

The inclination, θ, of the motion is such that

$$\tan\theta = \frac{v_y}{v_x} = \tan\alpha - \frac{32t}{v_0\cos\alpha}.$$

θ starts by being equal to α and diminishes continually.

The distance from the starting point to where the projectile crosses the horizontal plane through the former is known as the *range of the projectile.* With the starting point at the origin, the range is the value of x at the instant when y becomes zero. From (3) the range is thus found to be

$$R = \frac{v_0^2\sin\alpha\cos\alpha}{16} = \frac{v_0^2\sin 2\alpha}{32}. \quad (5)$$

Ex. 2. A particle is traveling along the circle $x^2 + y^2 = r^2$ with a constant speed of v_0 ft./sec. Find the horizontal and vertical components of its velocity when the particle is at the point (x, y) on the curve.

From the equation of the curve, we have on differentiating with respect to the time t,

$$2\,xx' + 2\,yy' = 0 \quad \text{or } xv_x + yv_y = 0,$$

where the primes indicate derivatives with respect to t. Besides, since the speed is given equal to v_0,

$$v_x^2 + v_y^2 = v_0^2.$$

Solving these two equations in v_x and v_y, we have

$$v_x = \pm\frac{v_0 y}{r},\ v_y = \mp\frac{v_0 x}{r}.$$

From the figure it is evident that $\frac{y}{r} = \sin\theta$, $\frac{x}{r} = \cos\theta$. So, the components of the velocity may also be written

$$v_x = \pm\, v_0 \sin\theta,\ v_y = \mp\, v_0 \cos\theta.$$

The ± sign appears because, when the speed alone is given, the direction of motion of the particle along the circle is not determined. If we add that the motion is to be counterclockwise, as indicated by the arrows in the figure, then the lower signs must be used, thus

$$v_x = -\frac{v_0 y}{r} = -v_0 \sin\theta,\ v_y = \frac{v_0 x}{r} = v_0 \cos\theta.$$

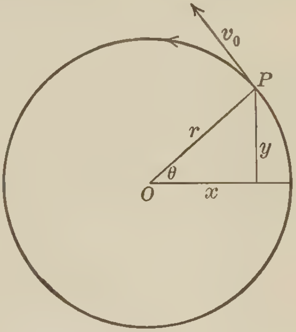

Fig. 33

Ex. 3. A particle is traveling along the straight line $3\,x - 4\,y = 5$. If the x-component of the velocity is 10 ft./sec., find the y-component and the velocity.

From the equation of the path

$$3\,x' - 4\,y' = 0, \text{ or } 3\,v_x - 4\,v_y = 0.$$

Since $v_x = 10,\ v_y = \frac{3}{4}\,v_x = \frac{15}{2}$ ft./sec., and the speed $\equiv |\,v\,| = \sqrt{v_x + v_y} = \frac{25}{2}$ ft./sec.

Since v_x and v_y are both positive, the motion along the path is to the right and upward.

Ex. 4. A particle is traveling toward the right along the upper half of the parabola $y^2 = 6\,x$ with a constant speed of 5 ft./sec. Find the horizontal and vertical components of its velocity when the particle is at the point (x_0, y_0) on the curve; also when it is at the point whose ordinate is 4.

Ex. 5. A particle is traveling along the straight line $x+y=5$ to the right and downward with a speed of 10 ft./sec. Find the horizontal and vertical components of its velocity.

Ex. 6. A particle is traveling in the first quadrant along the hyperbola $x^2-y^2=5$. If the vertical component of the velocity at any instant is inversely proportional to the ordinate of the position of the particle, find the horizontal component of the velocity and the speed when the particle is at (x_0, y_0). In particular if $v_y=4$ when $y=1$, find the speed when the particle is at (3, 2).

Ex. 7. Show that if a particle moves along the parabola $y=ax^2$ in such a way that the horizontal component of the velocity is inversely proportional to the abscissa of the position of the particle, the vertical component is constant.

Ex. 8. A particle moves around the ellipse $x^2+2y^2=24$ in either direction. At what points will its vertical and horizontal components be equal? At what points will they be equal numerically but of opposite signs?

26. Acceleration. In the case of rectilinear motion, *the acceleration is the time derivative of the velocity.* Thus, representing the acceleration by a, we have

$$a=D_tv=D_t^2s. \tag{1}$$

The algebraic sign of a, like that of v, has the physical significance of determining the sense or direction in which it is acting.

In the case of curvilinear motion, we make use of the fact that the velocity is given in terms of v_x and v_y, its projections on the coördinate axes. Acceleration like velocity is a vector quantity. We shall define it as the vector whose projections on the coördinate axes, a_x and a_y, are the time derivatives of v_x and v_y respectively. Thus

$$a_x = D_t v_x = D_t^2 x = x'', \quad a_y = D_t v_y = D_t^2 y = y''; \tag{2}$$

whence

$$\left.\begin{aligned} &\textit{amount of acceleration} \equiv |a| = +\sqrt{a_x^2 + a_y^2}, \\ &\textit{inclination of} \\ &\quad \textit{acceleration} \equiv \theta = \cos^{-1}\frac{a_x}{|a|} = \sin^{-1}\frac{a_y}{|a|} = \tan^{-1}\frac{a_y}{a_x}. \end{aligned}\right\} \tag{3}$$

If the motion of the particle is given in the form

$$x = f(t), \quad y = \phi(t),$$

we have

$$a_x = f''(t) = x'', \; a_y = \phi''(t) = y''$$

$$\therefore \; |a| = \sqrt{[f''(t)]^2 + [\phi''(t)]^2} = \sqrt{x''^2 + y''^2}. \tag{4}$$

While a_x and a_y determine the acceleration, another set of projections is frequently found of service. These are the projections of the acceleration upon the tangent and normal to the path of the particle at any instant. In Fig. 34 let OA represent the acceleration at the instant when the particle is at P. Let PT and PN be the corresponding tangent and normal to the path.

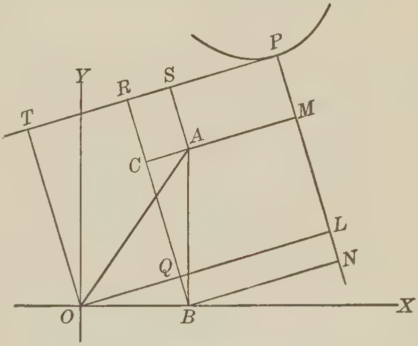

FIG. 34

The projection a_t of OA on PT is

$$a_t = TS = OQ + CA.$$

In the right triangle OBQ,

$$OB = a_x = x'',$$

$$\angle BOQ = \text{inclination of tangent} = \cos^{-1}\frac{v_x}{|v|} = \cos^{-1}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right).$$

$$\therefore\ OQ = OB\cos BOQ = \frac{x'x''}{\sqrt{x'^2+y'^2}}.$$

In the right triangle CBA,

$$AB = a_y = y''.$$

$$\angle ABC = \text{inclination of tangent} = \sin^{-1}\frac{v_y}{|v|} = \sin^{-1}\left(\frac{y'}{\sqrt{x'^2+y'^2}}\right).$$

$$\therefore\ CA = AB\sin ABC = \frac{y'y''}{\sqrt{x'^2+y'^2}}.\quad \text{Hence,}$$

$$a_t = \frac{x'x''+y'y''}{\sqrt{x'^2+y'^2}}. \tag{5}$$

The projection of OA on PN, the normal to the path, is, calling it a_n,

$$a_n = LM = BC - BQ.$$

As before, in the right triangle CBA,

$$AB = a_y = y''.$$

$$\angle ABC = \text{inclination of tangent} = \cos^{-1}\frac{v_x}{v} = \cos^{-1}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right).$$

$$\therefore\ BC = AB\cos ABC = \frac{x'y''}{\sqrt{x'^2+y'^2}}.$$

And as before, in the right triangle OBQ,

$$BO = OB\sin BOQ = \frac{y'x''}{\sqrt{x'^2+y'^2}}.\quad \text{Hence}$$

$$a_n = \frac{x'y''-y'x''}{\sqrt{x'^2+y'^2}} = \frac{x'y''-y'x''}{|v|}. \tag{6}$$

Remark. — Noting that $\frac{x'x''+y'y''}{\sqrt{x'^2+y'^2}} = D_t\sqrt{x'^2+y'^2} = D_t|v|$, we see from (5) that a_t is that part of the acceleration that produces the change in speed. In § 71 we shall see that $s' = \sqrt{x'^2+y'^2}$. Hence $a_t = s''$. The effect of a_n is to produce a change in the direction of the velocity. Later (§ 77) we shall see an interesting dynamical interpretation of the value of a_n. Also see (10) below.

In the case of the particle traveling in a circle with constant speed, Ex. 2, § 25, we found

$$v_x = x' = -\frac{v_0 y}{r}, \quad v_y = y' = -\frac{v_0 x}{r}.$$

From the equation of the path, $x^2 + y^2 = r^2$, we have, on differentiating,

$$xx' + yy' = 0 \text{ and } xx'' + yy'' + x'^2 + y'^2 = 0,$$

or
$$xx'' + yy'' = -v_0^2. \tag{7}$$

But, since the speed is constant, we have from the above Remark

$$x'x'' + y'y'' = 0; \; i.e. \; -\frac{v_0 y}{r}x'' + \frac{v_0 x}{r}y'' = 0,$$

or
$$-yx'' + xy'' = 0. \tag{8}$$

Solving (7) and (8) for x'' and y', we find

$$x'' = -\frac{v_0}{r}x \text{ and } y'' = -\frac{v_0}{r}y. \tag{9}$$

Hence, since any two mutually perpendicular diameters of the circle may be taken as the axes of rectangular coördinates, we see that *in the case of motion of a particle with constant speed along a circle, the component of the acceleration in any direction is proportional to the projection of the radius vector upon a diameter in that direction, and of opposite algebraic sign.*

Moreover, substituting the values of x', y', x'', y'' in (6), we find

$$a_n = \frac{v_0^2}{r}. \tag{10}$$

Ex. 1. Show that if x and y are any linear functions of t, the path of motion is a straight line, the velocity is constant, and the acceleration is zero.

Ex. 2. If $x = (t+1)^2$, $y = (t-1)^2$, find the velocity and the acceleration at any time t. Show that the path is a parabola.

Ex. 3. If $x=\dfrac{2a}{1+t^2}$, $y=\dfrac{2at}{1+t^2}$, find the velocity, the acceleration, and the path.

Ex. 4. If $x=at$, $y=\dfrac{a}{t}$, find the velocity, the acceleration, and the path.

Ex. 5. Show that if a particle moves along the parabola $y^2=4ax$ in such a way that the vertical component is constant, the acceleration is constant.

27. Angular velocity; angular acceleration. If a body such as a wheel or a pendulum moves about a fixed axis, each point of the body moves in a circle, whose center lies in the axis and whose plane is perpendicular to the axis. The position of the body at any instant is determined by the position of any one point of it (not in the axis) at that instant.

Fixing our attention on some point P, let θ be the angle (measured in radians) which OP, the radius of the circle described by P, makes with a fixed direction OX. The motion of the body will be known completely if θ is given as a function of t, the time; thus $\theta=f(t)$.

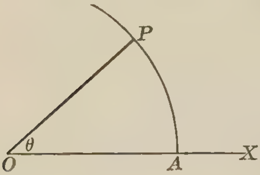

FIG. 35

We define $D_t\theta$ as the *angular velocity* of the body, and shall usually designate it by ω, thus

$$\omega=D_t\theta=f'(t)=\theta'. \tag{1}$$

Again $D_t^2\theta$ is defined as the *angular acceleration* of the body. Designating it by α, we have

$$\alpha=D_t^2\theta=D_t\omega=f''(t)=\theta''. \tag{2}$$

The algebraic signs of ω and α determine whether these are acting in the direction of increasing θ or in the opposite sense.

For the motion of P, a point of the body at a distance r from the axis, we note that if we let s be the length of the path described in the time t [in the figure we shall suppose that P is at A when $t = 0$],

$$s = r\theta. \tag{3}$$

Then, since r is a constant, we have for the velocity along the tangent

$$v = D_t s = rD_t\theta = r\omega, \tag{4}$$

and for the tangential acceleration (Remark, § 26)

$$a_t = D_t^2 s = rD_t^2\theta = r\alpha. \tag{5}$$

Ex. If the angle through which a rotating body, starting from rest, has moved at the end of t seconds is $\theta = 54\,t - 9\,t^2$, find its angular velocity and angular acceleration. Discuss the motion. Describe it at the end of 1 second; 3 seconds; 5 seconds; 10 seconds.

Evidently $\omega = 54 - 18\,t$, $\alpha = -18$, a constant.

Here $\omega = 0$ when $t = 3$, which means that, at the end of 3 seconds, the body is at rest instantaneously. After this instant ω is negative, which means that the body rotates in the opposite direction.

$\theta = 0$ for $t = 0$ and $t = 6$; which means that the body is again in its initial position at the end of the 6th second.

For $t = 1$, $\theta = 45$ radians, $\omega = 36$ rad./sec.
$t = 3$, $\theta = 81$ radians, $\omega = 0$.
$t = 5$, $\theta = 45$ radians, $\omega = -36$.
$t = 10$, $\theta = -360$ radians, $\omega = -126$.

EXERCISES

Find the angular velocity and angular acceleration and discuss the motion of a rotating body if the angle θ through which it has turned at the end of t seconds is given as the following functions of t:

1. $\theta = 80\,t - 10\,t^2$. **2.** $\theta = 30\,t^2 - 2\,t^3$. **3.** $\theta = 25\,t + 2\,t^2$.

4. $\theta = 6 - 8\,t + t^2$. What does the presence of the constant term 6 mean?

28. Other applications of the derivative. We have had occasion to make use of the fact that the time derivatives of certain quantities have a physical significance. Thus for rectilinear motion (§§ 25, 26),

$$D_t s = v, \quad D_t v = a,$$

and for a revolving body (§ 27),

$$D_t \theta = \omega, \quad D_t \omega = \alpha.$$

Correspondingly the derivatives of other quantities in Physics play important rôles. It is not practicable to give an adequate treatment of these here. They naturally arise in the detailed study of the various subjects involving them. But simply to illustrate the wide application of the Calculus in many branches of the physical sciences, a few illustrations are given below. The student is not expected to memorize them.

If, in an electric circuit, q is the charge or quantity of electricity, and t is the time, the time derivative of q is defined as the *current* i; i.e.

$$D_t q = i.$$

Again if L, a constant, is the coefficient of self-inductance of a coil in the circuit, the *induced electromotive force* E is given by

$$L D_t i = -E.$$

If Q is the quantity of heat required to raise the temperature of a unit mass of a substance from a standard temperature, say 0° C. to τ° C., the *specific heat* of the substance at τ° C. is

$$D_\tau Q = c.$$

If a force F applied to a rod of unit length and unit cross-section, in the direction of its length, produces a change of length s, then

$$D_s F = E, \text{ Young's } \textit{modulus of elasticity.}$$

If the effect of raising the temperature of a rod of unit length at 0° C. to τ° C. is to increase its length by l units, l being a function of τ, then

$$D_\tau l = C$$

is known as the *coefficient of linear expansion for the temperature τ.*

We shall see also, from time to time, that the derivatives of some geometrical configurations with respect to certain coördinates or lengths have a geometrical significance.

Ex. 1. Noting that if m is the mass of the particle,

$$\begin{aligned} &\text{momentum} &&\equiv M = mv, \\ &\text{force} &&\equiv F = ma, \\ &\text{kinetic energy} &&\equiv T = \tfrac{1}{2}mv^2, \end{aligned}$$

show that

$$D_t M = F; \quad D_v T = M; \quad D_t T = Fv.$$

Ex. 2. If A is the area of a circle, C its circumference, and r its radius, show that $D_t A = C$.

Ex. 3. If V is the volume of a sphere, S its surface, and r its radius, show that $D_r V = S$.

CHAPTER V

INTEGRATION

29. Anti-derivative. Integral. Heretofore we have started with a given function of a variable and then passed to its derivative. The inverse problem of finding a function when its derivative is known is involved in many important types of problems.

For example, in § 22 we saw that the length of the subnormal of the curve $y = f(x)$ is equal to yy'. Given y as a function of x, we can find its derivative and then calculate the length of the subnormal for any given point of the curve. But suppose we set this problem: Find the equation of a curve for which the length of the subnormal has the same value a for all points of the curve. This requires the finding of y as a function of x such that $yy' = a$. Noting that $D_x y^2 = 2\,yy'$ or $D_x\left(\frac{y^2}{2}\right) = yy'$, and $D_x(ax) = a$, we see that the relation $yy' = a$ results from differentiating both members of $\frac{y^2}{2} = ax$ and equating the results. Evidently the parabola $y^2 = 2\,ax$ has the required property.

Again, if we ask for a curve such that the slope of the tangent at any point is proportional to the square of the abscissa of the point, the requirement is that we find y as such a function of x that $y' = kx^2$, where k is the factor of proportionality. Since $D_x(kx^3) = 3\,kx^2$, it follows that $D_x\left(\frac{kx^3}{3}\right) = kx^2$, and the required equation of the curve is $y = \frac{k}{3}x^3$.

Suppose, on the other hand, that in the case of the rectilinear motion of a given body, the acceleration, due to the force acting, is a known function of the time, say $f(t)$. In the symbolism of § 26, this fact may be expressed

$$a = D_t v = f(t).$$

The velocity can then be expressed as a function of the time if we can find a function of t whose derivative is $f(t)$.

A function $F(t)$ whose derivative is a given function $f(t)$, i.e. $D_t F(t) = f(t)$, is defined as the *anti-derivative* or *integral* of $f(t)$ and may be designated by $D_t^{-1} f(t)$ or more simply by $D^{-1} f(t)$. From this definition, we have readily

$$D^{-1}x^2 = \frac{x^3}{3},\quad D^{-1}(2\,t) = t^2,\quad D_x^{-1} yy' = \frac{y^2}{2},\quad D_x^{-1}\frac{u'}{\sqrt{u}} = 2\sqrt{u},$$

as may be verified at once by differentiating the anti-derivative.

Remark. — In the symbol D^{-1}, the superscript " -1 " indicates an inverse operation just as it does in the case of the inverse trigonometric functions $\sin^{-1} x$, $\tan^{-1} x$, etc.

The process of finding the anti-derivative or integral of a function is known as *integrating* the function. Later we shall have occasion to use the terms " integration " and " integral " for an apparently different operation and its result, respectively. We shall, however, show that the dual use of the terms is consistent. But the symbolism suggested by the second type of operation is so convenient and so generally adopted, it seems desirable to introduce it (*i.e.* the symbolism) at this point, and then use it exclusively. In order to do this, it is necessary to make a digression, and explain what is meant by a " differential."

30. Differential. From the definition of the derivative of y with respect to x,

$$y' = \underset{\Delta x \longrightarrow 0}{L} \frac{\Delta y}{\Delta x},$$

it follows that the function

$$\frac{\Delta y}{\Delta x} - y'$$

approaches zero as Δx does; that is, it is an infinitesimal whenever Δx is. Representing it by η, we have, on clearing of fractions,

$$\Delta y = y'\Delta x + \eta\Delta x. \tag{1}$$

For small values of Δx, η is also a small quantity, and their product $\eta\Delta x$ is very small in comparison with either.

Let α and β be two infinitesimals connected by some relation so that their ratio $\frac{\beta}{\alpha}$ approaches a definite limit as both approach zero.

> If this limit is 0, β is said to be an *infinitesimal of higher order than* α;
> if this limit is a finite number different from 0, β is said to be an *infinitesimal of the same order as* α;
> if this limit is $\pm\infty$, β is said to be an *infinitesimal of lower order than* α.

Evidently $\eta\Delta x$ is an infinitesimal of higher order than Δx, for the ratio $\frac{\eta\Delta x}{\Delta x}$ has zero for limit as Δx approaches zero. The term $y'\Delta x$ in (1), which differs from Δy by an infinitesimal of higher order than Δx, is known as the *principal part* of Δy, and is called the *differential* of y. Using "d" for differential, we thus have defined the differential of $y = f(x)$ as

$$dy = y'\Delta x = f'(x)\Delta x; \tag{2}$$

i.e. the differential of a function of a variable is equal to the product of the derivative of the function and the increment of the variable.

In particular, if $y = x$, since $x' = D_x x = 1$,

$$dx = \Delta x; \tag{3}$$

that is, *the differential of the independent variable is the same as its increment.*

Combining (2) and (3) we shall write

$$dy = y' dx = f'(x)\, dx. \tag{4}$$

From (4) follows the important relation

$$y' = f'(x) = \frac{dy}{dx}. \tag{5}$$

This form of representing the derivative is very convenient and is largely used.

Attention should be called to the fact that $\frac{dy}{dx}$ is actually a fraction. The student should not let this statement mislead him into the error of confusing $\underset{\Delta x \to 0}{L}\left(\frac{\Delta y}{\Delta x}\right)$ with $\frac{\underset{\Delta x \to 0}{L} \Delta y}{\underset{\Delta x \to 0}{L} \Delta x}$. The latter expression has no meaning, since it is actually $\frac{0}{0}$. Moreover, from their definition, dy is not $\underset{x \to 0}{L} \Delta y$ and dx is not $\underset{x \to 0}{L} \Delta x$.

The following geometrical interpretation is both interesting and instructive. In the figure let

P be the point (x, y) on the curve $y = f(x)$,

Q be the point $(x + \Delta x, y + \Delta y)$ on the curve,

PT be the tangent to the curve at P.

$PR = \Delta x = dx$. Then

$RQ = \Delta y$, and $RT = dy$, because $\frac{RT}{PR} =$ slope of tangent at P.

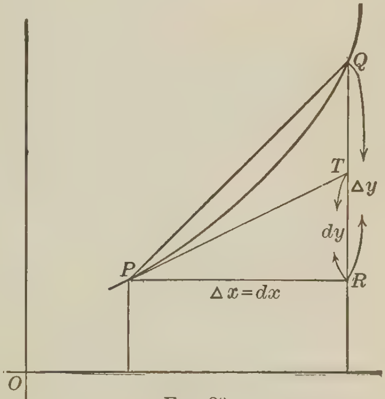

FIG. 36

It is worth noting that if the coördinates of P are

(x, y), those of Q are $(x+\Delta x,\ y+\Delta y)$ and those of T are $(x+dx,\ y+dy)$; that is, the point $(x+\Delta x,\ y+\Delta y)$ is on the curve $y=f(x)$, while $(x+dx,\ y+dy)$ is on the tangent to the curve at P.

The term "differentiating" a function $y=f(x)$ is used for the process of finding either its derivative or its differential. If the latter is desired, it is necessary to find the former and then multiply this by the differential of the variable. Thus, for example,

$$\text{if } y = x^2 + \frac{1}{x}, \qquad dy = 2\,x\,dx - \frac{dx}{x^2};$$

$$\text{if } y = \sqrt{x^2-1}, \quad dy = \frac{x\,dx}{\sqrt{x^2-1}}.$$

It is interesting to note that from the formula VI (§ 11) for differentiation of a function of the independent variable, we have

$$df(u) = D_u f(u) D_x u\,dx = D_u f(u)\,du = f'(u)\,du, \tag{6}$$

where the prime indicates differentiation with respect to u, the only variable in evidence. Thus we see that *the differential of a function of any variable* (either dependent or independent) *is the derivative of the function with respect to that variable multiplied by the differential of that variable.* Thus, for example,

$$\text{if } y = uv, \qquad dy = v\,du + u\,dv,$$

$$\text{if } y = \frac{u}{v}, \qquad dy = \frac{v\,du - u\,dv}{v^2}.$$

Find the differentials of the following functions:

1. $y = x^3 - 3\,x + 2.$

2. $y = \dfrac{\sqrt{x}}{\sqrt{x+1}}.$

3. $s = \dfrac{t}{\sqrt{t^2-1}}.$

4. $y = (u^2-1)^{\frac{3}{2}}.$

5. $z = \sqrt{\dfrac{y^2+1}{y^2-1}}.$

6. $z = x^2y^2.$

31. Differential notation for higher derivatives. Corresponding to what we may distinguish as the differential notation of the derivative, $\frac{dy}{dx}$ for y' or $D_x y$ [(5) § 30], the following analogous notation for higher derivatives also is in common use:

$$\frac{d^2y}{dx^2} \text{ for } y'' \text{ or } D_x^2 y,\ \frac{d^3y}{dx^3} \text{ for } y''' \text{ or } D_x^3 y,\ \cdots,\ \frac{d^ny}{dx^n} \text{ for } y^n \text{ or } D_x^n y. \quad (1)$$

Thus, in the case of rectilinear motion, we may write (§ 26)

$$\text{acceleration} \qquad \equiv a = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

and in the case of a rotating or revolving body (§ 27)

$$\text{angular acceleration} \equiv \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

The expressions in (1) also can be looked upon as fractions. This follows at once from the introduction of differentials of higher order by the definitions

$$d^2y = d(dy) = y''\,dx^2,\ d^3y = y'''\,dx^3,\ \cdots,\ d^ny = y^{(n)}\,dx^n. \qquad (2)$$

But no theorem analogous to that for differentials of the first order, contained in (6), § 30, holds for differentials of higher order. For the values of the latter depend upon the choice of independent variable. Thus, for example,

$$\text{if } y = x^3,\ \ d^2y = 6\,x\,dx^2.$$

Introducing a new variable u,

$$x = u^2,\ dx = 2\,u\,du, \text{ and } d^2y = 6\,u^2 \cdot 4\,u^2\,du^2 = 24\,u^4\,du^2.$$

If the change of variable is made before differentiating,

$$y = u^6, \text{ and } d^2y = D_u^2 y\,du^2 = 30\,u^4\,du^2.$$

Notwithstanding this fact, differentials of higher order have their applications. But we shall not employ them in this book. We shall regard the expressions

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \cdots,\ \frac{d^ny}{dx^n}$$

not as fractions, but as another notation for the corresponding derivatives $\qquad D_x^2 y,\ D_x^3 y,\ \cdots,\ D_x^n y.$

32. Approximate value of increment of function. Relative error. Recalling formula (1) of § 30,

$$\Delta y = y' \Delta x + \eta \Delta x, \tag{1}$$

we note that when Δx is small, η also will be small and their product $\eta \Delta x$ may be neglected in looking for an approximate value of Δy. The principal part $y' \Delta x$ is, therefore, a good approximation of the actual change in the function due to a small change in the variable.

Thus, for example, if $y = x^2$, find an approximate change in y as x grows from 3 to 3.1. Since $y' = 2x$, the principal part of $\Delta y = 2 \times 3 \times 0.1 = 0.6$. The actual value of $\Delta y = (3.1)^2 - 3^2 = 0.61$. The principal part differs from the actual value by 0.01, an error of 1.6% of Δy and 10% of the value of Δx.

The principal part of the increment of the function or, what is the same thing, the differential of the function can be taken as a convenient approximation in finding the error in the function corresponding to a small error in the measured or estimated value of the variable. Moreover, we shall be interested in finding the *relative error* $\frac{\Delta y}{y}$ of y, corresponding to an error Δx in x or to a relative error $\frac{\Delta x}{x}$. $\frac{dy}{y}$ is usually a good approximation for $\frac{\Delta y}{y}$.

Thus suppose that, in measuring the side of a square, an error of 2% is made; that is, $\frac{\Delta x}{x} = 0.02$, whence $\Delta x = 0.02\,x$. What are the resulting error and relative error of the corresponding value of the area of the square?

If A is the area corresponding to the true value of x, the error in A due to the error in x is approximately

$$dA = 2\,x \Delta x = 0.04\,x^2 = 0.04\,A.$$

$$\therefore \frac{dA}{A} = 0.04 \text{ or } 4\%.$$

Corresponding to any error, Δx, in x, we have

$$\frac{dA}{A}=\frac{2\,x\,\Delta x}{x^2}=\frac{2\,\Delta x}{x}.$$

That is, the relative error in A is approximately twice that in x for small values of Δx.

Ex. 1. Find approximately the relative error in the value of the volume of a cube if the error in measuring an edge of the cube is Δx.

Ex. 2. Find approximately the relative error in the value of the area of a circle if the error in measuring its diameter is Δx.

Ex. 3. Find approximately the relative error in the value of the volume of a sphere if the error in measuring its diameter is Δx.

Ex. 4. Boyle's law for the relation between the pressure and volume of a perfect gas, at a constant temperature, is $pv = c$, where c is a constant. If there is a slight error Δp in noting the pressure, what, approximately, is the corresponding relative error in the calculated volume?

Ex. 5. From the formula for kinetic energy $T=\frac{1}{2}mv^2$ find an expression for the relative error in T due to an error Δv in v.

Ex. 6. If $y = ax^m$, show that, approximately, the relative error in y is m times that in x. Compare this fact with the results in the earlier exercises of this section.

Ex. 7. Find an approximate value of $y=\sqrt{\dfrac{x+4}{x-4}}$ when $x = 5.03$; 4.97; 4.99; 5.02.

HINT. — The value of y for $x = 5$ is easily obtained. Use $y'\Delta x$ as an approximate value for Δy corresponding to $\Delta x = 0.03$.

Ex. 8. Find an approximate value of $y=\dfrac{x}{\sqrt{x^2+5}}$ when $x = 2.1$; also when $x = 1.98$.

33. Integration symbol. In § 29 we introduced the notion of integration, the process inverse to that of differentiation. Thus, if $D_x F(x) = f(x)$, we said that $F(x)$ is an integral or anti-derivative of $f(x)$, and the process of finding $F(x)$ when $f(x)$ is given is known as that of integration. In § 30 we introduced a new symbolism for derivative. Thus the relation between $F(x)$ and $f(x)$ may be written

$$\frac{dF(x)}{dx} = f(x), \quad \text{or}$$

$$dF(x) = f(x)dx. \tag{1}$$

From this form, we see that we may define $F(x)$ as an integral of $f(x)$ if its differential equals $f(x)dx$. Using the symbol of integration "$\int$," which thus designates an operator inverse to the differential operator "d," we shall write

$$F(x) = \int f(x)dx. \tag{2}$$

Differentiation is a direct process, that is, there is a general method for it (given in § 6) applicable to all functions. Integration is not so simple in principle. In the next three sections we shall consider some of the more obvious rules, reserving for a later chapter a more detailed study of the subject.

34. General principles. Constant of Integration. In § 10 we saw

V. $\quad D_x[F(x) + \Phi(x)] = D_xF(x) + D_x\Phi(x).$

If $F(x)$ is an integral of $f(x)$ and $\Phi(x)$ is an integral of $\phi(x)$, we may write V in the form

$$D_x[F(x) + \Phi(x)] = f(x) + \phi(x);$$

whence, by the definition of an integral, using the notation of § 33,

$$F(x) + \Phi(x) = \int [f(x) + \phi(x)]\,dx,$$

or, reversing the order of the terms and writing $F(x)$ and $\Phi(x)$ as integrals,

$$\int [f(x) + \phi(x)]\, dx = \int f(x)dx + \int \phi(x)dx. \tag{1}$$

We have thus established the important

Rule I. *An integral of the sum of two functions is obtained by taking the sum of integrals of the functions.*

This rule can be extended readily to apply to the algebraic sum of any finite number of functions.

In particular if $\Phi(x) = c$, *any* constant, its derivative $\phi(x) = 0$. Then by the definition of an integral, *any constant is an integral of zero.*

Combining this with Rule I we have

Theorem I. *If $F(x)$ is an integral of $f(x)$, then also $F(x) + c$, where c is any constant, is an integral of $f(x)$.* This constant is called the *constant of integration.*

Thus we see that while the derivative of a given function is a definite function, the same is not true of its integral; for the latter involves an arbitrary or undetermined constant of integration. Moreover, we can prove

Theorem I′. *If $F(x)$ is an integral of $f(x)$, $F(x) + c$ is the general form of integral of $f(x)$.*

For, if $\Phi(x)$ is any other integral of $f(x)$, we have

$$D_x[\Phi(x) - F(x)] = f(x) - f(x) = 0.$$

$$\therefore \Phi(x) - F(x) = c, \text{ or } \Phi(x) = F(x) + c.$$

Another of the formulas of § 10, which gives rise to a corresponding one for integrals, is

IV. $$D_x[cF(x)] = cD_xF(x).$$

If, as before, $F(x)$ is an integral of $f(x)$, IV may be written

$$D_x[c\,F(x)] = cf(x)\,; \text{ whence } cF(x) = \int cf(x)dx,$$

or reversing the order of the terms,

$$\int cf(x)dx = cF(x) = c\int f(x)dx. \qquad (2)$$

Hence

Rule II. *A constant factor may be transferred from one side of the symbol* $\int$ *to the other without affecting the result.*

35. Integral of x^n. Most formulas of differentiation furnish corresponding ones of integration. Thus from

$$Dx^n = nx^{n-1}$$

(III, § 10), it follows that

A.
$$\int x^m dx = \frac{x^{m+1}}{m+1} + c,$$

provided $m \neq -1$, as may be seen by differentiating the right-hand member. This rule may be expressed thus:

To find an integral of x^m, where m is any number except -1, increase the exponent of x by unity and divide by the new exponent. Since the exponent m in **A** corresponds to $n-1$ in III, § 10, we are not surprised that **A** fails to hold for $m = -1$. The integral for this case, viz. $\int \frac{dx}{x}$, will be found later.

EXERCISES

Find the following integrals, using formula **A** and the principles of § 34. In each case add the arbitrary constant. Verify the results by differentiating them.

1. $\int \frac{(x-1)^2}{\sqrt{x}}dx = \int \frac{x^2 - 2x + 1}{\sqrt{x}}dx = \int (x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx$

$$= \tfrac{2}{5}x^{\frac{5}{2}} - \tfrac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c.$$

2. $\int (x^2 - 2x)dx.$

3. $\int \frac{dx}{x^2} = \int x^{-2}dx.$

4. $\int 3\sqrt{x}\, dx.$

5. $\int 2(x^2 - 1)^2 dx.$

6. $\int (3x^2 - 4x + 4)dx.$

7. $\int (x^{\frac{1}{3}} + x^{-\frac{1}{3}})dx.$

8. $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\, dx.$

9. $\int (x^{\frac{4}{3}} + x^{-\frac{4}{3}})^2\, dx.$

10. $\int 2(x - 1)^3\, dx.$

11. $\int \sqrt{2ax}\, dx.$

12. $\int (\sqrt{a} - \sqrt{x})^2\, dx.$

36. Introduction of new variable. From the more general formula for differentiation of a power of a function (III′, § 11), $D_x u^n = nu^{n-1} D_x u$, we have, on passing to differentials,

$$du^n = nu^{n-1}u'dx = nu^{n-1}du \qquad (1)$$

where $u' = D_x u$. This suggests at once the integration formula

A′. $$\int u^m u' dx = \int u^m du = \frac{u^{m+1}}{m+1} + c,$$

provided $m \neq -1$, as may be verified by differentiating both members of the equation.

This important formula is frequently of practical value. The following exercises will illustrate:

Ex. 1. Evaluate $\int x\sqrt{x^2 + a^2}\, dx.$

Since $d(x^2 + a^2) = 2x\,dx$, let $x^2 + a^2 = u$. Differentiating, we have $2x\,dx = du$ or $x\,dx = \frac{1}{2}du$, and $\sqrt{x^2 + a^2} = \sqrt{u} = u^{\frac{1}{2}}$.

$$\therefore \int x\sqrt{x^2 + a^2}\, dx = \frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \tfrac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{(x^2 + a^2)^{\frac{3}{2}}}{3} + c.$$

Ex. 2. Evaluate $\int \frac{(3x+1)dx}{\sqrt{3x^2+2x-1}}$.

Since $d(3x^2+2x-1)=(6x+2)dx=2(3x+1)dx$,

let $3x^2+2x-1=u$.

Then $(6x+2)dx=du$ or $(3x+1)dx=\frac{1}{2}du$,

and $\frac{1}{\sqrt{3x^2+2x-1}}=u^{-\frac{1}{2}}$.

$$\therefore \int \frac{(3x+1)dx}{\sqrt{3x^2+2x-1}}=\frac{1}{2}\int u^{-\frac{1}{2}}du=\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c=u^{\frac{1}{2}}+c$$
$$=\sqrt{3x^2+2x-1}+c.$$

Ex. 3. Evaluate $\int \frac{dx}{\sqrt{(ax+b)^3}}$.

Since $d(ax+b)=a\,dx$, let $ax+b=u$. Then

$$dx=\frac{du}{a},\text{ and }\frac{1}{\sqrt{(ax+b)^3}}=u^{-\frac{3}{2}}.$$

$$\therefore \int \frac{dx}{\sqrt{(ax+b)^3}}=\frac{1}{a}\int u^{-\frac{3}{2}}\,du=\frac{1}{a}\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}+c=-\frac{2}{au^{\frac{1}{2}}}+c$$
$$=\frac{-2}{a\sqrt{ax+b}}+c.$$

EXERCISES

Evaluate the following integrals:

1. $\int (x+1)(x^2+2x-5)^3\,dx$.
2. $\int \sqrt{1-x}\,dx$.
3. $\int \frac{x^2-2}{\sqrt{x^3-6x+4}}\,dx$.
4. $\int (ax+b)^{\frac{3}{2}}\,dx$.
5. $\int (1-2x)^3\,dx$.
6. $\int x\sqrt{3-2x^2}\,dx$.
7. $\int \frac{2\,dx}{\sqrt[3]{2x+3}}$.
8. $\int \frac{x\,dx}{(2x^2-5)^3}$.

9. $\int x\sqrt{x^2-4}\,dx.$

10. $\int \frac{6\,x\,dx}{(3\,x^2-4)^2}.$

11. $\int x^2\sqrt{6-3\,x^3}\,dx.$

12. $\int \frac{x\,dx}{\sqrt{ax^2+b}}.$

13. $\int \frac{dx}{\sqrt{2\,x-1}}.$

14. $\int \sqrt[3]{a+bx}\,dx.$

15. $\int \frac{x^2dx}{(2-x^3)^2}.$

37. Geometric applications. We have noted frequently that the slope of the tangent to a curve $y=f(x)$ is equal to the value of the derivative y' at the point of contact. Moreover, the equations of tangent and normal to the curve, the lengths of tangent, normal, subtangent, and subnormal all involve the derivative. Hence any characterization of the curve by means of any of these will, when expressed analytically, involve the derivative y'. The determination of the equation of the curve, that is, the finding of y as a function of x, will involve integration or the finding of an integral of y'.

Several illustrations of this kind of problem were given in § 29. Let us discuss these again.

Ex. 1. Find the equation of the curve for which the length of the subtangent has the same value a for all points of the curve.

From the given facts we have

$$yy'=a.$$

Introducing the differential notation, this may be written

$$y\frac{dy}{dx}=a \text{ or } y\,dy=a\,dx.$$

The last form states that the differentials of two functions are equal. Hence the original functions are either equal or differ by a constant; that is,

$$\frac{y^2}{2}=ax+c \tag{1}$$

where c is any constant.

This illustrates the general fact, that since the integral is indeterminate, involving a constant of integration, any definition of a curve involving this derivative necessarily defines a whole family of curves. Frequently a special curve is singled out by some further restriction upon it. Thus, if a parabola of the family (1) is required to pass through the origin, substituting $x = 0$ and $y = 0$ in equation (1), we see that $c = 0$; that is, $y^2 = 2ax$ is the special curve of the family which passes through the origin.

If it is to pass through (1, 2), we have, on putting $x = 1$ and $y = 2$ in (1),

$$2 = a + c. \quad \therefore c = 2 - a.$$

Hence $y^2 = 2ax + 4 - 2a$ is the special curve of the family which passes through (1, 2).

Ex. 2. Find the equation of the family of curves such that the slope of the tangent at any point is equal to three times the square of the abscissa of the point. In particular, find the curve of the family which passes through the point (2, 1).

From the given facts we have

$$y' = 3x^2 \text{ or } dy = 3x^2 dx.$$

$$\therefore y = \int 3x^2\,dx = x^3 + c,$$

where c is any constant.

In particular, that curve of the family passing through (2, 1) will be given when c satisfies the relation $1 = 2^3 + c$ or $c = -7$; the equation of the required curve is thus $y = x^3 - 7$.

Ex. 3. Find the equation of the family of curves such that the part of the normal included between the coördinate axes is bisected by the curve.

From (5) and (6), § 22, it follows that our family of curves satisfies the requirements if

$$y\sqrt{1+y'^2}=\frac{x}{y'}\sqrt{1+y'^2}, \text{ or}$$

$$yy'=x, \text{ or } y\,dy=x\,dx.$$

$$\therefore \frac{y^2}{2}=\frac{x^2}{2}+c, \text{ or } y^2-x^2=2\,c, \text{ or } x^2-y^2=-2\,c.$$

Since c is *any* constant, $-2\,c$ also is *any* constant and may be replaced by a single letter, say k. So we may finally write the equation of the family of curves as

$$x^2-y^2=k, \tag{2}$$

where k is any constant. For positive values of k, we have a family of equilateral hyperbolas whose transverse axes are horizontal. If k is negative, the equilateral hyperbolas have their transverse axes vertical. For all values of k, including 0 (in which case we have a pair of lines), the corresponding curves (2) satisfy the requirements of the problem.

It will be left as an exercise for the student to find the particular values of k determining the special curves through (5, 3); (4, 5); (3, − 3); (− 13, 12); (0, 0), respectively.

Ex. 4. Find the equation of the family of curves in which the subnormal at any point on the curve is proportional to the abscissa of the point.

Ex. 5. Find the equation of the family of curves in which the subnormal at any point on the curve is equal to the square of the abscissa of the point.

Ex. 6. Find the equation of the family of curves in which the slope of the tangent to the curve at any point (x, y) is equal to $\frac{x}{y}$.

Ex. 7. Find the equation of the family of curves in which the tangent to the curve at any point is perpendicular to the line joining the point of contact with the origin.

Ex. 8. Find the equation of the family of curves in which the tangent to the curve at any point is perpendicular to the line joining the point of contact with the fixed point (a, b).

Ex. 9. Find the equation of the family of curves such that the triangle formed by the tangent to the curve, the ordinate to the point of contact, and the axis of x has to a constant area.

Ex. 10. Find the equation of the family of curves such that the subtangent is equal to the cube of the ordinate to the point of contact. Find the particular curve which passes through $(-2, 3)$.

38. Area under a curve. In the figure let LM be the curve $y = f(x)$. We shall endeavor to find the area enclosed by the curve, the axis of x, and the ordinates at $x = a$ and $x = b$. This is usually referred to as the *area under the curve from* $x = a$ *to* $x = b$, even if part or all of the curve is below the axis of x. It may be considered as generated by a moving ordinate as it sweeps from its initial position at $x = a$ to its final position at $x = b$.

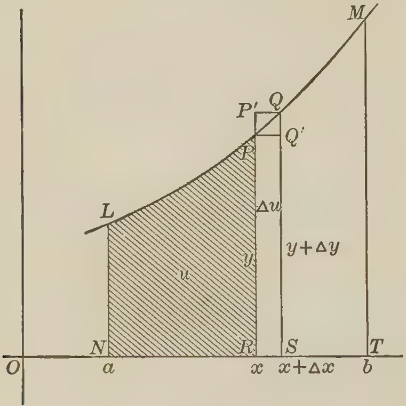

FIG. 37

Let u be the variable area swept over by the moving ordinate. It is evidently a function of x; for $x = a$, $u = 0$; for $x = b$, $u = A$, the desired area.

Let (x, y) be P, some point on the curve intermediate between L and M, and $(x + \Delta x, y + \Delta y)$ a neighboring point Q on the curve. Suppose that y', the derivative, is of the same sign all along the curve, then Δy will have the same sign. (This restriction on the curve will be removed later, in Remark 1, below.) In the case illustrated in Fig. 37, Δy is positive. Here $u =$ area $NLPR$ and $\Delta u =$ area $RPQS$. The latter is greater than $RPQ'S$ and less than $RP'QS$; that is,

$$y\Delta x < \Delta u < (y + \Delta y)\Delta x; \text{ whence}$$

$$y < \frac{\Delta u}{\Delta x} < y + \Delta y.$$

If the curve were descending, these inequality signs would be reversed.

We are supposing y to be a continuous function of x. Hence as Δx approaches zero, Δy does also and $y + \Delta y$ approaches y. Hence

$$\underset{\Delta x \to 0}{L} \frac{\Delta u}{\Delta x} \equiv D_x u = y = f(x). \tag{1}$$

We have thus shown that, *if* u *is the area under the curve* $y = f(x)$ *from* $x = a$ *to an ordinate corresponding to a variable* x, *the derivative of* u *with respect to* x *is equal to* $f(x)$. Introducing the differential notation, (1) may be written

$$du = y\,dx = f(x)\,dx. \tag{1'}$$

Let $F(x)$ be an integral of $f(x)$. Integrating (1′), we have

$$u = \int f(x)\,dx = F(x) + c.$$

With the original ordinate $x = a$ known, the value of c is determined. For, as already noted, when $x = a$, $u = 0$.

$\therefore\ 0 = F(a) + c$, whence $c = -F(a)$, and

$$u = F(x) = -F(a).$$

Finally, the area under the curve from $x = a$ to $x = b$ is then

$$\text{Area} = F(b) - F(a), \tag{2}$$

where $F(x) = \int f(x)dx$.

REMARK 1. In the deduction of the formula for the area under a curve the restriction was imposed that the slope of the tangent to the curve must be constant in sign as x varies from a to b. That formula (2) holds for other continuous curves can be seen readily, as follows:

If the derivative is not constant in sign, the curve has maximum and minimum points. Break up the curve into portions ending at such maximum or minimum points. The area under the entire curve is, of course, the sum of the areas under each of the partial curves. Expressing these by (2) and adding, the result follows. Thus, for example, in Fig. 38, let C be a maximum point and D a minimum. Consider now the areas under AC, CD, and DB; their sum is evidently the area under the whole curve AB. If the abscissas of C and D are c and d respectively,

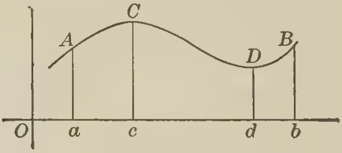

FIG. 38

$$\text{area under } AC = F(c) - F(a),$$
$$\text{area under } CD = F(d) - F(c),$$
$$\text{area under } DB = F(b) - F(d).$$

Adding, we have

$$\text{area under } AB = F(b) - F(a),$$

as before.

REMARK 2. Any function $F(x)$ was defined as an integral of $f(x)$, *i.e.* $F(x) = \int f(x)dx$, if $DF(x) = f(x)$. Such a function is said to be an *indefinite* integral of $f(x)$. As in (2), we shall have frequent occasion to use the difference between the values which some indefinite integral, $F(x)$, takes for two values of the variable, say, $x = a$ and $x = b$. On this account we shall make use of the convenient notation

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a). \tag{3}$$

The integral $\int_a^b f(x)dx$ is known as *the definite integral of x, taken between the limits a and b.* The order in which these limits are taken is essential; a is known as the *lower limit* of the integral, and b the *upper limit.* Since (§ 34) any indefinite integral of $f(x)$ differs from any other one by an additive constant only, it is clear that the value of the definite integral given by (3) is independent of the choice of the indefinite integral $F(x)$.

The use of the symbol $F(x)\Big]_a^b$ for the difference $F(b) - F(a)$, taken in the definite order, will be found of practical value.

Remark 3. Attention should be called to the fact that, in the deduction of formula (2), *it was assumed that $f(x)$ is finite for all values of x from $x = a$ to $x = b$.* A single illustration will suffice to show how this formula may be misleading when $f(x)$ becomes infinite somewhere in the interval from $x = a$ to $x = b$. Thus, in the case of the curve $y = \frac{1}{x^2}$, y becomes infinite for $x = 0$. The area under the curve from $x = -1$ to $x = 1$ is given by (2) as $-\frac{1}{x}\Big]_{-1}^{1} = -1 - (1) = -2$. As a matter of fact, the area under the curve is infinite.

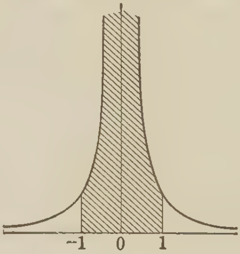

Fig. 39

While, in certain cases, formula (2) gives the correct area even when $f(x)$ becomes infinite for some value of x in the interval from $x = a$ to $x = b$, the fact that it may lead to error suggests the caution that, for the present, formula (2) be used *only* in case $f(x)$ is finite throughout the interval of integration. Later, the more general case will be considered.

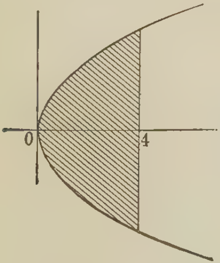

Fig. 40

Ex. 1. Find the area between the parabola $y^2 = 4x$ and the line $x = 4$.

Because of symmetry with respect to the x-axis, the required area is twice the area under the curve $y = 2\sqrt{x}$ between the origin and the ordinate $x = 4$. Hence the required area is equal to

$$2\int_0^4 2\sqrt{x}\,dx = 4\int_0^4 x^{\frac{1}{2}}dx = \tfrac{8}{3}x^{\frac{3}{2}}\Big]_0^4 = \tfrac{8}{3}(4)^{\frac{3}{2}} - 0 = \tfrac{64}{3}.$$

Ex. 2. Find the area between the curves $y^2 = x$ and $y = x^2$. The curves intersect in the origin and the point (1, 1).

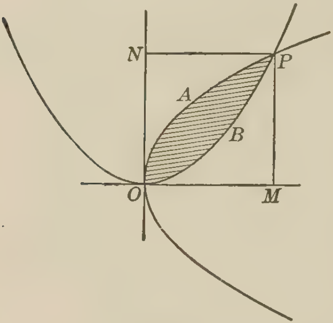

FIG. 41

The area between them is the difference between the areas under each of the curves from the origin to the ordinate $x = 1$.

Area

$$OAPM = \int_0^1 \sqrt{x}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big]_0^1 = \tfrac{2}{3}.$$

Area

$$OBPM = \int_0^1 x^2\,dx \quad = \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3}.$$

Hence

the required area $= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

It will thus be noted that *in the square ONPM, the shaded area is equal to each of the unshaded ones.*

Ex. 3. Find the area between the curve $y = x^3$, the axis of y, and $y = 8$. Evidently $B = (2, 8)$.

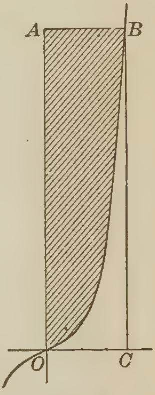

FIG. 42

The required area is equal to that of the rectangle $OABC$ minus the area under the curve from the origin to the ordinate $x = 2$. For the latter

$$\text{Area } OBC = \int_0^2 x^3\,dx = \frac{x^4}{4}\Big]_0^2 = 4.$$

Moreover, Area $OABC = 16$.

Hence the required area $OABO = 12$.

Ex. 4. Find the area between the semi-cubical parabola $y^3 = 8\,x^2$ and the line $y = 2\,x$.

Ex. 5. Find the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3$ and the coördinate axes.

Ex. 6. Sketch the curve $x^2 + y^{\frac{1}{2}} = 4$ and find the area under it from $x = 0$ to $x = 2$; also between $x = -2$ and $x = 2$.

Ex. 7. Sketch the curve $y^2 = x^2(4 - x^2)$ and find its total area.

Ex. 8. Sketch the curve $y^3 = x^3(9 - x^2)$ and find the area under it between $x = 0$ and $x = 3$; also between $x = -3$ and $x = 3$. Explain the result.

Ex. 9. Find the area of the loop of the curve $y^2 = x^4(1 - x^3)$.

Ex. 10. Find the area under $y = x(x^2 - 4)^2$ between $x = 0$ and $x = 2$; also between $x = -2$ and $x = 2$.

39. Physical applications. Vertical projectile. As a first application, let us consider the motion of a projectile thrown vertically upward with an initial velocity of v_0 ft./sec. Disregarding the resistance of the air, the only force acting is that of gravity, which produces an acceleration vertically downward of $-g$ ft./sec². For numerical purposes it will be sufficiently accurate to take $g = 32$.

Using the notation of § 31

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = -g. \tag{1}$$

Integrating, we have

$$\frac{ds}{dt} = v = -gt + c, \tag{2}$$

where c is a constant. To determine it we make use of the fact that $v = v_0$ when $t = 0$.

$$\therefore\ v_0 = 0 + c, \quad \text{or} \quad c = v_0.$$

Hence, for all values of t,

$$\frac{ds}{dt} = v = -gt + v_0. \tag{3}$$

Integrating again,

$$s = -\tfrac{1}{2}gt^2 + v_0t + c, \tag{4}$$

where c is a constant. If the initial position of the projectile is at a distance s_0 from the point used as the origin from which to measure s, then $s = s_0$ when $t = 0$. Using this in (4), it follows that $c = s_0$. Hence the position of the projectile at any instant t is given by

$$s = -\tfrac{1}{2}gt^2 + v_0t + s_0. \tag{5}$$

The distance of the projectile at any time t from its initial position is given by

$$s - s_0 = -\tfrac{1}{2}gt^2 + v_0t. \tag{5'}$$

The projectile will reach the highest point in its path at a time t for which s is a maximum. To find this put

$$\frac{ds}{dt} = -gt + v_0 = 0. \quad \therefore \; t = \frac{v_0}{g}.$$

Since $\dfrac{d^2s}{dt^2} = -g$, a negative quantity, this value of t renders s a maximum. (As we should expect from the nature of the problem, the velocity is zero at this instant.)

The maximum height is

$$\frac{v_0^2}{2g} + s_0, \text{ or } \frac{v_0^2}{2g} \text{ above the starting point.} \tag{6}$$

From (5) or (5′) it follows readily that $s = s_0$ for $t = 0$ and $t = \dfrac{2v_0}{g}$. For the second value of t, $v = -v_0$. Hence we see that *it takes the projectile the same length of time to fall from its highest point to its original position as it does to rise to its highest point, and it reaches its initial position with its initial speed, but with its direction of motion reversed.*

In the special case when the body simply falls from rest, subject to gravity only, it is convenient for our immediate purpose to take the starting point as the origin from which to measure s. The differential equation[1] of motion is, as before,

$$\frac{dv}{dt} = -g. \tag{1}$$

---

[1] A differential equation is one that involves differentials or derivatives.

Introducing s as a new variable, using VI, § 11, and noting that $\frac{ds}{dt} = v$,

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

Hence the differential equation of motion (1) may be put in the form

$$v\frac{dv}{ds} = -g, \text{ or } 2\,v\,dv = -2\,g\,ds. \qquad (7)$$

The factor 2 has been introduced on both sides to avoid fractions after integration. Integrating (7),

$$v^2 = -2\,gs + c. \qquad (8)$$

Since we selected our origin so that $s = 0$ when $v = 0$, it follows that $c = 0$ in (8). Hence

$$v = \sqrt{-2\,gs}, \qquad (9)$$

a very important formula in Mechanics. Since s is negative, the quantity under the radical sign in (9) is positive.

The student, as an exercise, should obtain (9) from (2) and (5) above, after putting $s_0 = 0,\ v_0 = 0$.

In the general case show that

$$v^2 = 2\,g(s_0 - s) + v_0^2. \qquad (10)$$

EXERCISES

1. A man at the edge of a cliff 80 feet high throws a ball upward with a speed of 64 ft./sec. How high above the cliff does it rise? When will it reach the foot of the cliff, and with what velocity?

2. With what speed must a body be thrown vertically upward from a tower 192 feet high in order that it may strike the ground at the end of 6 seconds? How high will it rise? With what speed will it strike the ground?

3. With what speed must a body be thrown vertically downward from a tower 192 feet high in order to reach the ground at the end of 2 seconds? With what speed will it strike the ground? Compare results with those of Ex. 2, and explain.

4. A body is dropped from rest at the edge of a cliff. Find its velocity and position at the end of 1 second; 3 seconds; 4 seconds; 5 seconds. Two seconds later a second body is projected from the same place vertically downward with a speed of 96 ft./sec. Will it overtake the first body? If so, when, and with what speed does it pass it?

If the second body is projected with the same speed as before, but three seconds after the first, what can be said of their ever meeting? Explain.

5. A body is dropped from rest at the edge of a cliff and at the same instant a second body is projected vertically upward with a speed of v_0 ft./sec. For what value of v_0 will the second body overtake the first one? Explain your result.

6. A stationary balloon at an elevation of 1800 feet drops a bomb. At the same instant, from a point directly below, a projectile is fired vertically upward with an initial speed of 450 ft./sec. When and where does the projectile meet the bomb?

40. Projectile, general case. Suppose the projectile to start from the origin with an initial speed v_0, making an angle α with the horizontal plane. The components of the initial velocity are $v_0 \cos \alpha$ in the direction of the x-axis and $v_0 \sin \alpha$ in the direction of the y-axis.

Since gravity acts vertically downward, there is no horizontal acceleration. Hence, for the horizontal component of the motion,

$$\frac{d^2x}{dt^2} = 0.$$

$\therefore \dfrac{dx}{dt} = v_x = c$, where c is a constant. To determine this we note that $v_x = v_0 \cos \alpha$ when $t = 0$. $\therefore c = v_0 \cos \alpha$. Hence for all values of t

$$\frac{dx}{dt} = v_x = v_0 \cos \alpha. \qquad (1)$$

Integrating again, we have

$$x = (v_0 \cos \alpha)t + c.$$

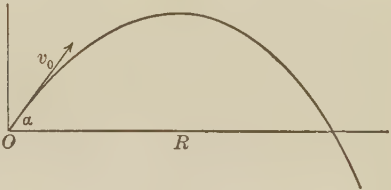

FIG. 43

Since $x = 0$ when $t = 0$, because the projectile started at the origin, we have

$$0 = 0 + c, \text{ or } c = 0.$$

$$\therefore \boldsymbol{x = (v_0 \cos \alpha)t} \qquad (2)$$

gives the x-component of the motion.

For the y-component, we note that the acceleration is vertically downward and g in amount. Hence

$$\frac{d^2y}{dt^2} = -g.$$

Integrating, we have

$$\frac{dy}{dt} = v_y = -gt + c.$$

Since $v_y = v_0 \sin \alpha$ when $t = 0$, we have

$$v_0 \sin \alpha = 0 + c.$$

$$\therefore \frac{dy}{dt} = v_y = -gt + v_0 \sin \alpha. \qquad (3)$$

Integrating again,

$$y = -\tfrac{1}{2} gt^2 + (v_0 \sin \alpha)t + c.$$

Since $y = 0$ when $t = 0$, $c = 0$; and we have for the y-component of the motion

$$\boldsymbol{y = -\tfrac{1}{2} gt^2 + (v_0 \sin \alpha)t.} \qquad (4)$$

Equations (2) and (4) determine the position of the projectile at any instant t. They are the equations of the path of the projectile, in parametric form. To find the rectangular coördinate equation, t must be eliminated between them resulting in

$$y = -\frac{g}{2\,v_0^2\cos^2\alpha}x^2 + (\tan\alpha)x, \tag{5}$$

a parabola. To find the elements of this parabola write (5) in the form

$$y = -\frac{g}{2\,v_0^2\cos^2\alpha}\left(x^2 - \frac{2\,v_0^2\sin\alpha\cos\alpha}{g}x\right).$$

Then complete the square on the right by adding

$$-\frac{g}{2\,v_0^2\cos^2\alpha}\cdot\frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2} \text{ or } -\frac{v_0^2\sin^2\alpha}{2\,g}$$

to both sides. The equation of the parabola then takes the form

$$y - \frac{v_0^2\sin^2\alpha}{2\,g} = -\frac{g}{2\,v_0^2\cos^2\alpha}\left(x - \frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 \text{ or}$$

$$\left(x - \frac{v_0^2\sin 2\,\alpha}{2\,g}\right)^2 = -\frac{2\,v_0^2\cos^2\alpha}{g}\left(y - \frac{v_0^2\sin^2\alpha}{2\,g}\right). \tag{5'}$$

The vertex of the parabola is at the point

$$\left(\frac{v_0^2\sin 2\,\alpha}{2\,g}, \frac{v_0^2\sin^2\alpha}{2\,g}\right); \tag{6}$$

the focus is at

$$\left[\frac{v_0^2\sin 2\,\alpha}{2\,g}, \frac{v_0^2}{2\,g}(\sin^2\alpha - \cos^2\alpha)\right]; \tag{7}$$

and the directrix is

$$y = \frac{v_0^2}{2\,g}. \tag{8}$$

It will be noted that the directrix is independent of α, the *angle of fire.* Its distance from the origin is the distance the projectile would rise if projected vertically upward, that is, when $\alpha = \frac{\pi}{2}$ [(6), § 39].

The highest point in the path of the projectile is reached when $\frac{dy}{dx} = 0$; that is, when, using (3), $t = \frac{v_0 \sin \alpha}{g}$. Putting this in (2) and (4), we see that the highest point is

$$\left(\frac{v_0^2 \sin 2\alpha}{2g}, \frac{v_0^2 \sin^2 \alpha}{2g}\right),$$

the vertex of the parabola, as already found in (6).

From (4) it is easy to see that y vanishes a second time when $t = \frac{2v_0 \sin \alpha}{g}$, that is, in twice the time it takes to attain its maximum value.

The distance from its initial position to the point where the projectile again crosses the horizontal plane through its starting point is known as the *range* of the projectile. As we have just seen, this second point is reached when $t = \frac{2v_0 \sin \alpha}{g}$. Hence the range is

$$R = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}. \tag{9}$$

Since $\sin 2\alpha$ assumes its greatest value when $2\alpha = \frac{\pi}{2}$, that is, when $\alpha = \frac{\pi}{4}$ or 45°, a projectile fired with a given speed v_0 will have its *maximum range* $\frac{v_0^2}{g}$ when the angle of fire is 45°. Throughout this discussion we are not taking account of the air resistance.

In Fig. 44 let OQ represent the section of a plane making an angle ϕ with the horizontal plane. The distance from the initial point O to Q, where the projectile strikes the inclined plane, is the *range of the projectile along this plane.* Designate it by R_ϕ. To find its length, we proceed to find the coördinates of Q, the point of intersection of the path of the projectile

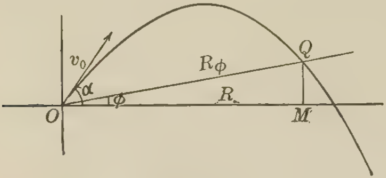

FIG. 44

$$y = -\frac{g}{2\,v_0^2\cos^2\alpha}x^2 + (\tan\alpha)\,x, \tag{5}$$

and the line OQ, whose equation is

$$y = (\tan\phi)\,x. \tag{10}$$

Combining these, we find for the x of Q

$$OM = \frac{2\,v_0^2\cos^2\alpha}{g}(\tan\alpha - \tan\phi) = \frac{2\,v_0^2\cos\alpha\sin(\alpha-\phi)}{g\cos\phi}.$$

It is not necessary to find the y of Q. For, noting that

$$OM = OQ\cos\phi, \text{ or } OQ = \frac{OM}{\cos\phi}, \text{ we have}$$

$$R_\phi = \frac{2\,v_0^2\cos\alpha\sin(\alpha-\phi)}{g\cos^2\phi}. \tag{11}$$

This evidently reduces to the value of R given by (9) when

$$\phi = 0.$$

EXERCISES

1. A gun is fired at an angle of 30° with an initial speed of 1024 ft./sec. What are its range and time of flight? How high does it go?

2. Answer the questions in Ex. 1, when the angle of fire is 60°.

3. Show that for the same initial speed the range and time of flight are the same when the angle of fire is α as when it is $\frac{\pi}{2} - \alpha$.

4. A cannon has a maximum range of 20 miles. What is the muzzle speed of the shell? When fired at an angle of 45°, how high does it go, and what is the time of flight?

5. Show that when the angle of fire is 45°, the height attained by the projectile is one quarter the range.

6. What must be the initial speed of a tennis ball, struck at a negligible distance from the ground, which shall have a range of 160 feet, and shall not rise more than 20 feet?

7. A boy standing on the roof of a house 36 feet above the ground wishes to throw a ball to a boy on the ground across a street 60 feet wide. If he throws the ball in a horizontal direction, what speed must he give it?

8. Show that the speed of a projectile at any point in its path is equal to the speed a body would acquire if it fell, starting at rest, from the directrix of the path to that point, under the action of gravity alone.

9. A boy can give a ball an initial speed of 80 ft./sec. What is the farthest point of the ceiling of a gymnasium 20 feet high that he can hit with the ball, neglecting the height of his hand when the ball is thrown? At what angle must he throw it?

10. A man at the edge of a vertical cliff 100 feet high, overlooking a lake, hits a golf ball, giving it an initial speed of 120 ft./sec. and making an angle of 30° with the horizontal plane. How far from the foot of the cliff does the ball strike the water? With what speed and at what angle does it strike? How much longer does it travel in point of time, and

how much farther from the base of the cliff does it go than it would have done if the man had been on a level with the water?

11. Show that the foci of the various paths of a projectile thrown from a point O with a given speed v_0 in all possible directions lie on a sphere whose center is O.

12. Show that if a shell, either allowed to drop vertically or fired at an angle, explodes in the course of its motion, the various particles of it continue to move in such a way that at each instant they are on a sphere having for center the position the shell would have occupied if it had not exploded. [Here the sizes of the shell and of its parts after explosion are neglected. In other words, each is supposed to be replaced by a particle located at its center of mass.]

CHAPTER VI

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

41. Derivative of sin x. If $y = \sin x$, $y + \Delta y = \sin(x + \Delta x)$.

$\therefore\ \Delta y = \sin(x + \Delta x) - \sin x = 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$,[1] and

$$\frac{\Delta y}{\Delta x} = 2\cos\left(x + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\Delta x}. \qquad (1)$$

To find the limit of the right-hand member of (1) as Δx approaches zero, we proceed as follows: In Fig. 45, AB is the arc of the circle subtending the angle θ at the center. The angle θ is supposed to be measured in radians, not degrees. If the radius is r, we have

$BC = r\sin\theta$, $BA = r\theta$,

$DA = r\tan\theta$, $OC = r\cos\theta$.

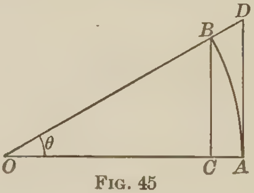

FIG. 45

The area of the

triangle $OBC = \frac{1}{2}r^2\sin\theta\cos\theta$,

sector $OBA = \frac{1}{2}r^2\theta$,

triangle $ODA = \frac{1}{2}r^2\tan\theta$.

Evidently, from the figure,

$$\tfrac{1}{2}r^2\sin\theta\cos\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2\tan\theta$$

Dividing by $\frac{1}{2}r^2\sin\theta$, we have

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}, \text{ or}$$

$$\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta.$$

---

[1] This is a direct application of the formula of trigonometry

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

This holds for all values of θ in the first quadrant. As θ is made smaller and smaller, the two extreme members of our inequalities both approach 1. But the value of $\frac{\sin\theta}{\theta}$ always lies between those of the extreme members. Hence its value also approaches 1 as a limit as θ approaches 0; *i.e.*

$$\underset{\theta\rightarrow 0}{L}\frac{\sin\theta}{\theta}=1. \tag{2}$$

This can now be applied to finding the limit in the case of (1). Evidently $\underset{\Delta x\rightarrow 0}{L}\cos\left(x+\frac{\Delta x}{2}\right)=\cos x.$ So we are concerned with the limit of $\frac{2\sin\frac{\Delta x}{2}}{\Delta x}$. The limit of this fraction is 1; for, on putting $\frac{\Delta x}{2}=\theta$, the fraction reduces to $\frac{\sin\theta}{\theta}$. Hence, we have finally on passing to the limit in (1)

X. $$\boldsymbol{D_x\sin x=\cos x,\ \text{or}\ d\sin x=\cos x\,dx;}$$

i.e. *the derivative of the sine of a variable is its cosine.* Combining this with VI, § 11, we have the very useful formula

X′. $$\boldsymbol{D_x\sin u=\cos u D_x u,\ \text{or}\ d\sin u=\cos u\,du.}$$

Thus, if $y=\sin x^2$ [here $u=x^2$],

$$y'=\cos x^2 D_x x^2=2x\cos x^2,\ \text{or}\ dy=2x\cos x^2 dx.$$

Again, if $y=\sin(ax+b)$,

$$y'=a\cos(ax+b),\ \text{or}\ dy=a\cos(ax+b)\,dx.$$

EXERCISES

Differentiate the following:

1. $y=\sin 2x.$
2. $y=x^2\sin x.$
3. $y=\frac{x-\sin x}{x+\sin x}.$
4. $y=\sqrt{\frac{1+\sin x}{1-\sin x}}.$
5. $y=\sin 2x\sin^2 x.$
6. $y=\sin 3x\sin^3 x.$

7. $y = \sin nx \sin^n x.$

8. $y = \sqrt{1 - \sin^2 x}.$

9. $y = \dfrac{\sin x}{\sqrt{1 - \sin^2 x}}.$

10. $y = \dfrac{\sqrt{1 - \sin^2 x}}{\sin x}.$

11. $y = \dfrac{1}{\sqrt{1 - \sin^2 x}}.$

12. $y = \dfrac{1}{\sin x}.$

42. Derivatives of the other trigonometric functions. To find the derivative of $\cos x$, we note that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and use X′. Thus

$$D_x \cos x = D_x \sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - x\right) D_x\left(\frac{\pi}{2} - x\right) = - \sin x.$$

Hence

XI. $\boldsymbol{D_x \cos x = - \sin x}$, or $\boldsymbol{d \cos x = - \sin x\, dx}$;

i.e. *the derivative of the cosine of a variable is minus its sine.* Combined with VI, § 11, we have

XI′. $\boldsymbol{D_x \cos u = - \sin u\, D_x u}$, or $\boldsymbol{d \cos u = - \sin u\, du}$.

To find the derivative of $\tan x$, we note that $\tan x = \dfrac{\sin x}{\cos x}$, and differentiate this as a quotient. Thus

$$D_x \tan x = D_x \frac{\sin x}{\cos x} = \frac{\cos x D_x \sin x - \sin x\, D_x \cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

Hence

XII. $\boldsymbol{D_x \tan x = \sec^2 x}$, or $\boldsymbol{d \tan x = \sec^2 x\, dx}$;

i.e. *the derivative of the tangent of a variable is the square of its secant.* Combined with VI, § 11, we have

XII′. $\boldsymbol{D_x \tan u = \sec^2 u\, D_x u}$, or $\boldsymbol{d \tan u = \sec^2 u\, du}$.

As before, making use of the fact either that $\cot x = \tan\left(\frac{\pi}{2} - x\right)$ or that $\cot x = \frac{\cos x}{\sin x}$, we have

XIII. $D_x \cot x = -\csc^2 x$, or $d \cot x = -\csc^2 x\, dx$;

i.e. *the derivative of the cotangent of a variable is minus the square of its cosecant;* and

XIII′. $D_x \cot u = -\csc^2 u\, D_x u$, or $d \cot u = -\csc^2 u\, du$.

Noting that $\sec x = \frac{1}{\cos x}$, the student will have no difficulty in showing that

XIV. $D_x \sec x = \tan x \sec x$, or $d \sec x = \tan x \sec x\, dx$;

i.e. *the derivative of the secant of a variable is the product of its tangent and secant;* and

XIV′. $D_x \sec u = \tan u \sec u\, D_x u$, or $d \sec u = \tan u \sec u\, du$.

Finally noting either that $\csc x = \sec\left(\frac{\pi}{2} - x\right)$ or that $\csc x = \frac{1}{\sin x}$, we have

XV. $D_x \csc x = -\cot x \csc x$, or $d \csc x = -\cot x \csc x\, dx$;
i.e. *the derivative of the cosecant of a variable is minus the product of its cotangent and cosecant;* and

XV′. $D_x \csc u = -\cot u \csc u\, D_x u$, or $d \csc u = -\cot u \csc u\, du$.

REMARK. — Several facts in connection with the derivatives of the trigonometric functions are worthy of note:

1. They are given in terms of the trigonometric functions, and in particular in terms of those with which the original functions are closely related by algebraic relations; thus in the expressions for their respective derivatives sine and cosine, tangent and secant, cotangent and cosecant pair off.

2. The derivatives of three of the functions (sine, tangent, and secant) appear with the plus sign, while those of the remaining ones, co-functions of the former (cosine, cotangent, and cosecant), involve the minus sign.

EXERCISES

Differentiate the following:

1. $y = \cos(ax + b)$.
2. $y = x \sin x + \cos x$.
3. $x = \tan^2 ax$.
4. $y = \sec^2 ax$.
5. $y = \cos^3 x - 3 \cos x$.
6. $y = x \csc x$.
7. $y = 2x \sin x + (2 - x^2) \cos x$.
8. $y = 2 \cos^2 \left(\frac{x}{2}\right)$.
9. $y = \cos^2 ax - \sin^2 ax$.
10. $y = \cot^2 ax - \csc^2 ax$.
11. $y = x \tan^2 x$.
12. $y = \sqrt{\frac{\sec x + \tan x}{\sec x - \tan x}}$.
13. $\rho = a\sqrt{\cos 2\theta}$.
14. $\rho = \frac{a}{1 - \cos\theta}$.
15. $\rho = a \sin^3 \left(\frac{\theta}{3}\right)$.
16. $\rho = a \sec^2 \left(\frac{\theta}{2}\right)$.

In the following curves, whose equations are given in terms of the parameter, t, find the slope $\frac{dy}{dx}$:

17. $x = a \cos t, \quad y = a \sin t$.
18. $x = a \cos^2 t, \quad y = a \sin^2 t$.
19. $x = a \cos^3 t, \quad y = a \sin^3 t$.
20. $x = a(t - \sin t), \quad y = a(1 - \cos t)$.
21. $x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t)$.

43. Inverse trigonometric functions. If $x = \sin y$, then y is, in turn, a function of x, and we shall write $y = \sin^{-1} x$, which is read "inverse sine of x" and means "the arc whose sine is x." The notation $y = \text{arc} \sin x$ also is in common use. But we shall adopt the former notation. Corresponding to a given value of y, the value of x is unique; x is said to be a *single-valued* function of y. But corresponding to a given value of x, we have an indefinite number of values of y; y is said to be a *many-valued* function of x. Fig. 46 contains a graph of this function. Evidently y is real when $-1 \leq x \leq 1$, that is, when x lies in the interval $(-1, 1)$, and imaginary for all other values. We can make y single-valued if we exclude all values greater than $\frac{\pi}{2}$, and all less than $-\frac{\pi}{2}$, that is, if we consider only such values of y which satisfy the relations $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, that is, which lie in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

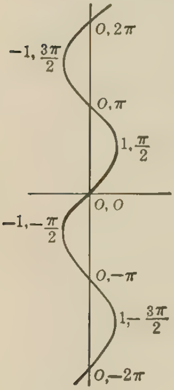

FIG. 46

This shall be supposed to be the case from now on. To find the derivative of $\sin^{-1} x$, we proceed as follows:

$$y = \sin^{-1} x. \quad \therefore x = \sin y;$$

$$\text{whence } D_y x = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Using VII, § 11, viz. $D_x y = \frac{1}{D_y x}$, we have $D_x y = \frac{1}{\sqrt{1 - x^2}}$, or

$$\textbf{XVI.} \quad D_x \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \text{ or } d \sin^{-1} x = \frac{dx}{\sqrt{1 - x^2}}; \text{ and}$$

$$\textbf{XVI}'. \quad D_x \sin^{-1} u = \frac{D_x u}{\sqrt{1 - u^2}}, \text{ or } d \sin^{-1} u = \frac{du}{\sqrt{1 - u^2}}.$$

From Fig. 46, it is seen that, restricting $\sin^{-1} x$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, its derivative is everywhere positive; hence, in XVI and XVI′, the positive sign must be used with the radical in the denominator.

To find the derivative of $\cos^{-1} x$, we proceed in an analogous manner, thus;

$y = \cos^{-1} x. \quad \therefore x = \cos y$; whence $D_y x = -\sin y$

$= -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$. Using VII, §11, we have

XVII. $D_x \cos^{-1} x = \dfrac{1}{-\sqrt{1 - x^2}}$, or $d \cos^{-1} x = \dfrac{dx}{-\sqrt{1 - x^2}}$; and

XVII′. $D_x \cos^{-1} u = \dfrac{D_x u}{-\sqrt{1 - u^2}}$, or $d \cos^{-1} u = \dfrac{du}{-\sqrt{1 - u^2}}$.

As illustrated in Fig. 47, $\cos^{-1} x$ is a many-valued function, if not restricted in value. We shall make it single-valued by restricting it to lie in the interval $(0, \pi)$. In this interval it is a continuously diminishing function, hence its derivative is always negative. This is consistent with the negative sign before the radical in XVII.

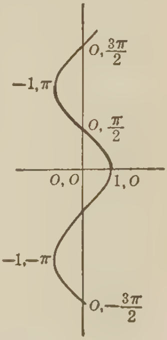

FIG. 47

REMARK 1. The fact that $D_x \cos^{-1} x = -D_x \sin^{-1} x$ follows at once from the relation $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$.

To find the derivative of $\tan^{-1} x$ we have $y = \tan^{-1} x. \therefore x = \tan y$; whence $D_y x = \sec^2 y = 1 + \tan^2 y = 1 + x^2$. Using VII, § 11, we have

XVIII. $D_x \tan^{-1} x = \dfrac{1}{1 + x^2}$, or $d \tan^{-1} x = \dfrac{dx}{1 + x^2}$; and

XVIII′. $D_x \tan^{-1} u = \dfrac{D_x u}{1 + u^2}$, or $d \tan^{-1} u = \dfrac{du}{1 + u^2}$.

As illustrated in Fig. 48, $\tan^{-1} x$ is many-valued. But it becomes single-valued if restricted to lie in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The student should have little difficulty in establishing the following formulas:

XIX. $D_x \cot^{-1} x = -\dfrac{1}{1 + x^2}$, or $d \cot^{-1} x = -\dfrac{dx}{1 + x^2}$.

XIX′. $D_x \cot^{-1} u = -\dfrac{D_x u}{1 + u^2}$, or $d \cot^{-1} u = -\dfrac{du}{1 + u^2}$.

XX. $D_x \sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$, or $d \sec^{-1} x = \dfrac{dx}{x\sqrt{x^2-1}}$.

XX′. $D_x \sec^{-1} u = \dfrac{D_x u}{u\sqrt{u^2-1}}$, or $d \sec^{-1} u = \dfrac{du}{u\sqrt{u^2-1}}$.

XXI. $D_x \csc^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$, or $d \csc^{-1} x = -\dfrac{dx}{x\sqrt{x^2-1}}$.

XXI′. $D_x \csc^{-1} u = -\dfrac{D_x u}{u\sqrt{u^2-1}}$, or $d \csc^{-1} u = -\dfrac{du}{u\sqrt{u^2-1}}$.

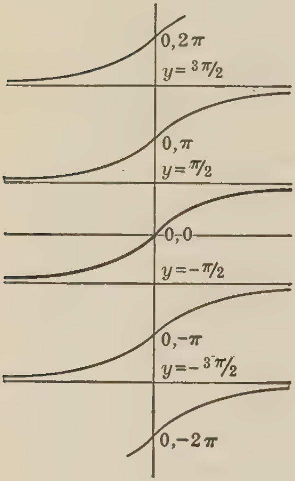

Fig. 48

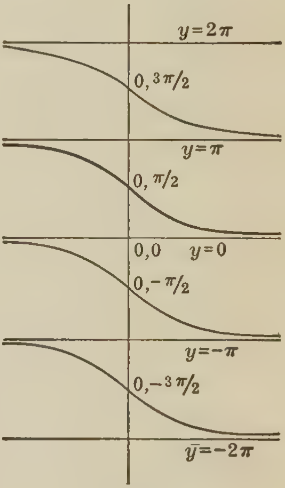

Fig. 49

As illustrated in Fig. 49, $\cot^{-1} x$ becomes a single-valued function if restricted to lie in the interval $(0, \pi)$. In this interval the function is continuously diminishing, a fact consistent with the negative sign in XIX. In the restricted intervals for $\tan^{-1} x$ and $\cot^{-1} x$ we have

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

In regard to $\sec^{-1} x$, the matter is not so simple. In order that the positive sign may be used for the radical in XX, we shall make $\sec^{-1} x$ single-valued by restricting it to the interval $\left(0, \frac{\pi}{2}\right)$ for x positive and to $\left(-\pi, -\frac{\pi}{2}\right)$ for x negative. See Fig. 50. The second interval could have been replaced by $\left(\pi, \frac{3\pi}{2}\right)$; but it is customary to use the smallest numerical values for our intervals. The fact that $\sec^{-1} x = \cos^{-1}\frac{1}{x}$ may be made to account for the discontinuity in the interval for $\sec^{-1} x$.

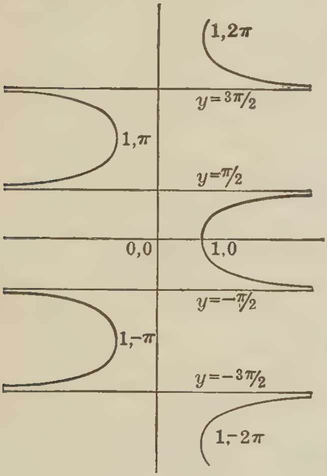

Fig. 50

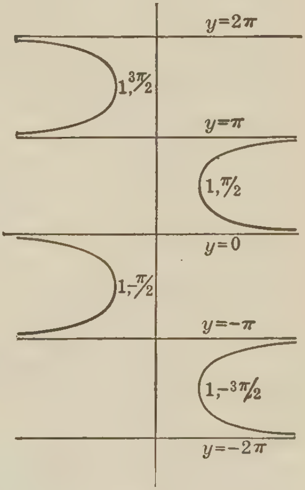

Fig. 51

In an analogous manner, the negative sign in XXI will be found to hold throughout if $\csc^{-1} x$ $\left(\text{which is equal to } \sin^{-1}\frac{1}{x}\right)$ is made single-valued by restricting it to the interval $\left(0, \frac{\pi}{2}\right)$ for x positive and to $\left(-\pi, -\frac{\pi}{2}\right)$ for x negative. See Fig. 51.

In the restricted intervals for $\sec^{-1} x$ and $\csc^{-1} x$

$$\sec^{-1} x + \csc^{-1} x = \begin{cases} \dfrac{\pi}{2} \text{ for } x \text{ positive.} \\ -\dfrac{3\,\pi}{2} \text{ for } x \text{ negative.} \end{cases}$$

Another formula that is found serviceable at times is that for the derivative of $\text{vers}^{-1} x$ (read "inverse versine of x" or "arc whose versine is x").

Since $\text{vers}\, x = 1 - \cos x$, $D_x \text{vers}\, x = \sin x$. If $y = \text{vers}^{-1} x$, $x = \text{vers}\, y$. $\therefore\ D_y x = \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - (1 - \text{vers}\, y)^2}$ $= \sqrt{2 \text{ vers } y - \text{vers}^2 y} = \sqrt{2\,x - x^2}$. Hence

$$\textbf{XXII.}\quad D_x \,\textbf{vers}^{-1} x = \frac{1}{\sqrt{2\,x - x^2}},$$

$$\text{or}\quad d\,\textbf{vers}^{-1} x = \frac{dx}{\sqrt{2\,x - x^2}}\,;\ \text{and}$$

$$\textbf{XXII}'.\quad D_x \,\textbf{vers}^{-1} u = \frac{D_x u}{\sqrt{2\,u - u^2}},$$

$$\text{or}\quad \textbf{d}\,\textbf{vers}^{-1} u = \frac{du}{\sqrt{2\,u - u^2}}.$$

From Fig. 52 $\text{vers}^{-1} x$ is seen to be single-valued in the interval $(0, \pi)$. In this interval its derivative is positive. Hence the positive sign of the radical is used.

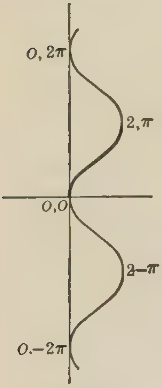

Fig. 52

Remark 2. Attention should be called to the fact that the derivatives of the inverse trigonometric functions are algebraic functions and not transcendental.

EXERCISES

Differentiate the following:

1. $y = \sin^{-1}\dfrac{x}{a}$.
2. $y = x \sin^{-1} x$.
3. $y = \cos^{-1}\dfrac{a}{x}$.
4. $y = \tan^{-1}\dfrac{x}{a}$.

5. $y = \sin^{-1}(\sin x)$.

6. $y = \cos^{-1}(\sin x)$.

. $y = \cot^{-1}\frac{x}{a} + \cot^{-1}\frac{a}{x}$.

8. $y = \sec^{-1}\frac{x}{a}$.

9. $y = \sec^{-1}\sqrt{x^2+1}$.

10. $y = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$.

11. $y = \cos^{-1}\sqrt{1-x^2}$.

12. $y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$.

13. $y = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$.

14. $y = \sqrt{a^2-x^2} + a\sin^{-1}\frac{x}{a}$.

15. $y = \sqrt{a^2-x^2} + x\sin^{-1}\frac{x}{a}$.

16. $y = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$.

17. $y = -x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$.

18. $y = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}$.

19. $y = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}$.

20. $y = -\frac{\sqrt{a^2-x^2}}{x} + \cos^{-1}\frac{x}{a}$.

21. $y = \csc^{-1}\frac{x}{a}$.

22. $y = \operatorname{vers}^{-1}\frac{x}{a}$.

23. $y = a\operatorname{vers}^{-1}\frac{x}{a} - \sqrt{2ax-x^2}$.

24. $y = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}$.

44. Logarithm. The student will recall that *the logarithm of a number, to a given base, is the exponent of that power of the base which is equal to the number;* thus, to say that $y = \log_a x$ means that $a^y = x$, or $a^{\log_a x} = x$. We shall presuppose that *a and x are both positive, and $a \neq 1$.*

The following properties of logarithms are of prime importance. They result from the definition of a logarithm.

1. $\log_a a = 1$; for $a^1 = a$.
2. $\log_a 1 = 0$; for $a^0 = 1$.

3. $\log_a xy = \log_a x + \log_a y$; for from

$$x = a^{\log_a x} \text{ and } y = a^{\log_a y},$$

we have on multiplication

$$xy = a^{\log_a x + \log_a y};$$

whence property 3 follows. In words, this says that *the logarithm of the product of two numbers is equal to the sum of their respective logarithms.*

4. $\log_a \frac{x}{y} = \log_a x - \log_a y$; for, on taking the quotient of the two relations above instead of their product, we have $\frac{x}{y} = a^{\log_a x - \log_a y}$; i.e. *the logarithm of the quotients of two numbers is equal to the difference of their respective logarithms.*

5. $\log_a x^n = n \log_a x$; for, raising each member of $x = a^{\log_a x}$ to the nth power, we have

$$x^n = a^{n \log_a x};$$

i.e. *the logarithm of the nth power of a number is equal to n times the logarithm of the number.*

Here n may be any number, positive or negative, integral or fractional.

The relation between the logarithms of the same number in two systems with the bases a and b respectively is readily obtained. Thus, for x any positive number $x = b^{\log_b x}$. Taking logarithms, with a as a base, of both members, we have,

6. $$\log_a x = \log_b x \log_a b,$$

a formula reminding us of VI, § 11.

A further property of logarithms follows from this formula by letting $x = a$; whence

7. $$1 = \log_b a \log_a b, \text{ or } \log_b a = \frac{1}{\log_a b},$$

which reminds us of VII, § 11.

A further preliminary fact must be emphasized before proceeding to the deduction of the formula for the derivative of a logarithm.

In § 41 we came across the function $\frac{\sin x}{x}$, which has no definite value for $x = 0$; but we saw that it approaches the perfectly definite value 1, as a limit, as x approaches 0. The function $(1 + x)^{\frac{1}{x}}$ also fails to take a definite value for $x = 0$, although it can be shown (but we shall not do so here) to approach a perfectly definite value, as a limit, as x approaches 0. For all other values of x inside the interval $(-1, \infty)$, *i.e.* for

$$-1 < x < \infty,$$

the function assumes definite values. Its graph is indicated in Fig. 53. $x = -1$ and $y = 1$ are vertical and horizontal asymptotes, respectively. The limit of $(1 + x)^{\frac{1}{x}}$ as x approaches 0 is designated by e, and has been calculated very accurately to a larger number of decimal places. It is approximately 2.7182818285.

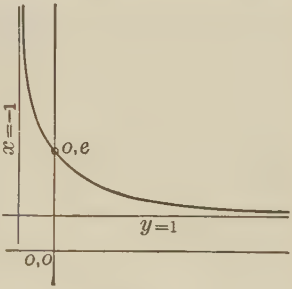

FIG. 53

The number e vies with π (the ratio of the circumference to the diameter of a circle) for the distinction of being the most important constant arising in Mathematics. That it is irrational or incommensurable has been known a long time. But that it is transcendental (*i.e.* not algebraic, by which is meant that it cannot be the root of any algebraic equation with integral coefficients) was first proved in 1875 by the French mathematician Hermite. It is interesting that this proof of Hermite suggested to the German mathematician Lindemann in 1882 a proof of the transcendence of π, which proof established for all time the impossibility of the ancient problem of squaring the circle by use of the straight-edge and dividers only. We shall see later (§ 146) that e can be represented by the infinite series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

A value correct to 8 decimal places will be given by taking 12 terms of the series and carrying these out to 9 places. Thus

$1 + 1 = 2$

$\frac{1}{2!} = 0.5$

$\frac{1}{3!} = 0.166\ 666\ 667$

$\frac{1}{4!} = 0.041\ 666\ 667$

$\frac{1}{5!} = 0.008\ 333\ 333$

$\frac{1}{6!} = 0.001\ 388\ 889$

$\frac{1}{7!} = 0.000\ 198\ 413$

$\frac{1}{8!} = 0.000\ 024\ 802$

$\frac{1}{9!} = 0.000\ 002\ 756$

$\frac{1}{10!} = 0.000\ 000\ 276$

$\frac{1}{11!} = 0.000\ 000\ 025$

$\frac{1}{12!} = 0.000\ 000\ 002$

The sum of these is 2.718 281 830. That this is correct to at least 7 places may be seen as follows: The error due to omitting the rest of the terms $\frac{1}{13!} + \frac{1}{14!} + \frac{1}{15!} + \cdots$ is evidently less than $\frac{1}{13!}\left(1 + \frac{1}{13} + \frac{1}{13^2} + \cdots\right)$; for $\frac{1}{14!} = \frac{1}{13!}\frac{1}{14} < \frac{1}{13!}\frac{1}{13}$, $\frac{1}{15!} = \frac{1}{13!}\frac{1}{14}\frac{1}{15} < \frac{1}{13!}\frac{1}{13^2}$, and so on. The sum of this geometric progression is $\frac{1}{13!}\frac{1}{1 - \frac{1}{13}} = \frac{1}{12!\,13}$, which is less than 0.000 000 000 2. This does not affect even the 9th decimal place. Allowing for errors not greater than unity in the 9th place in the values of the terms of the series, their aggregate cannot do more than affect the 8th place in the value of e. As a matter of fact the value found is correct to 8 places.

Starting now with $y = \log_a x$, we have $y + \Delta y = \log_a(x + \Delta x)$; whence, using property 4,

$$\Delta y = \log_a(x + \Delta x) - \log_a x = \log_a \frac{x + \Delta x}{x} = \log_a\left(1 + \frac{\Delta x}{x}\right).$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) = \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}},$$

because of property 5. In order to have this involve a function of the type $(1 + x)^{\frac{1}{x}}$, multiply and divide the right-hand member by x and make use of property 5. Then

$$\frac{\Delta x}{\Delta y} = \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

Now $\frac{\Delta x}{x}$ approaches zero with Δx. [Here Δx, as always in finding the derivative, is the variable, x being temporarily a constant; in this case, any positive constant, if we wish to avoid imaginary quantities, and, in any case, x is definitely not zero.] Passing to the limit[1] for $\Delta x \rightarrow 0$,

XXIII. $D_x \log_a x = \frac{1}{x} \log_a e$, or $d \log_a x = \frac{dx}{x} \log_a e$,

and using VI, § 11,

XXIII′. $D_x \log_c u = \frac{D_x u}{u} \log_a e = \frac{u'}{u} \log_a e$, or $d \log_a u = \frac{du}{u} \log_a e$.

In all that has gone before, no restriction has been placed upon a except that it may be a positive constant different from 1. For purposes of numerical computation $a = 10$ has been found most practical. The system of logarithms with 10 as a base is known as the *Briggs* or *common system*, and is the one with which the student has become familiar in Trigonometry and Algebra. But in the Calculus, where the derivative of a logarithm plays an important rôle, it is clear that there will be an immense simplification if the number e is taken as the bases. As a matter of fact this is what is done. This system of logarithms, with e as its base, is known as the *Naperian* or

---

[1] Here use is made of the general principle of limits that $\underset{\alpha \rightarrow \alpha_0}{L} \log f(\alpha) = \log \left[\underset{\alpha \rightarrow \alpha_0}{L} f(\alpha)\right]$ under certain restrictions, which exist in this case.

natural system. Unless otherwise stated, we shall confine ourselves to this system in what follows, and shall use the notation $\log u$ to stand for $\log_e u$.[1] With this notation we have

$$\textbf{XXIII}''. \qquad D_x \log x = \frac{1}{x}, \quad \text{or} \quad d \log x = \frac{dx}{x}, \text{ and}$$

$$\textbf{XXIII}'''. \qquad D_x \log u = \frac{D_x u}{u} = \frac{u'}{u}, \quad \text{or} \quad d \log u = \frac{du}{u};$$

i.e. *the derivative (differential) of the natural logarithm of a function equals the derivative (differential) of the function divided by the function.* The function $\frac{u'}{u}$, the derivative of $\log u$, is often referred to as the *logarithmic derivative* of u.

NOTE 1. In actual practice, it will frequently be simpler to make use of the properties of logarithms noted at the beginning of the section, before applying XXIII'''. Thus if

$$y = \log \sqrt{x^2 - 2x}, \text{ write } y = \tfrac{1}{2} \log (x^2 - 2x).$$

Then $\quad y' = \dfrac{1}{2}\,\dfrac{2x - 2}{x^2 - 2x} = \dfrac{x - 1}{x^2 - 2x}.$

Again if $\quad y = \log \sqrt{\dfrac{a - x}{a + x}}$, write $y = \tfrac{1}{2} \log (a - x) - \tfrac{1}{2} \log (a + x).$

Then $\quad y' = \dfrac{1}{2}\,\dfrac{-1}{a - x} - \dfrac{1}{2}\,\dfrac{1}{a + x} = -\dfrac{a + x + a - x}{2(a^2 - x^2)} = \dfrac{-a}{a^2 - x^2}.$

NOTE 2. At times, when it is required to differentiate a function having a number of factors, it is advisable first to take the logarithm of the function (simplifying this by means of properties 3, 4, 5), then differentiate, and finally multiply the result by the original function. Thus, for example, if

$$y = \frac{x^3 \sqrt{x^2 - 1}}{x^2 + 1}, \ \log y = 3 \log x + \tfrac{1}{2} \log (x^2 - 1) - \log (x^2 + 1), \text{ and}$$

$$\frac{y'}{y} = \frac{3}{x} + \frac{x}{x^2 - 1} - \frac{2x}{x^2 - 1} = \frac{2x^4 + 3x^2 - 3}{x(x^2 - 1)(x^2 + 1)}. \qquad \therefore y' = \frac{x^2(2x^4 + 3x^2 - 3)}{(x^2 + 1)^2 \sqrt{x^2 - 1}}.$$

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[1] In some works, especially in Engineering, the notation "ln u" is used for the natural logarithm of u.

Assuming that property 5 holds for n any real number, rational or irrational, we are now in a position to prove that the important formula III, § 10, holds for any real number n. For, if $y = x^n$, $\log y = n \log x$; whence on differentiating, $\frac{Dy}{y} = \frac{n}{x}$. $\therefore Dy = \frac{ny}{x} = nx^{n-1}$.

EXERCISES

Differentiate the following:

1. $y = \log(x^2 - 5x + 6)$.
2. $y = \log\sqrt{\frac{1+x}{1-x}}$.
3. $y = x \log x$.
4. $y = \log(\log x)$.
5. $y = \log(x + \sqrt{x^2 + a^2})$.
6. $y = \log(x - \sqrt{x^2 + a^2})$.
7. $y = \log \sin x$.
8. $y = \log \cos x$.
9. $y = \log \tan x$.
10. $y = \log \cot x$.
11. $y = \log \sec x$.
12. $y = \log \csc x$.
13. $y = \log(\sec x + \tan x)$.
14. $y = \log(\csc x - \cot x)$.
15. $y = x \tan^{-1} x - \log\sqrt{1+x^2}$.
16. $y = \frac{\sqrt{x}(x^2-1)^2}{\sqrt{(x^2+1)^3}}$.
17. $y = x\sqrt[3]{\frac{1-x}{1+x}}$.
18. $y = \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$.

45. Exponential. If $y = a^u$, where a is a constant and u is a function of x, we have, on taking the logarithm of both sides,

$$\log y = u \log a; \text{ whence, using XXIII}'''$$

$$\frac{D_x y}{y} = \log a \, D_x u. \quad \text{Hence}$$

XXIV. $D_x a^u = a^u \log a \, D_x u = a^u u' \log a$, or $da^u = a^u \log a \, du$.

In particular, if $a = e$, we have the important formula

XXIV′. $D_x e^u = e^u D_x u = e^u u'$, or $de^u = e^u du$,

which carries with it the special case

XXIV″. $D_x e^x = e^x$, or $de^x = e^x dx$.

The function e^u is spoken of as the *exponential function of* u or simply the *exponential of* u, and is sometimes designated by "exp. u." In words, formula XXIV′ says *the derivative (differential) of the exponential of u is equal to the exponential multiplied by the derivative (differential) of u.*

Remark. — The exponential and logarithm are inverse functions, just as the square and square root, sine and inverse sine, etc. Although for purposes of obtaining their respective derivatives, it was simpler for us to introduce the logarithm first and the exponential afterward, the exponential is really the simpler function in many respects. Its derivative is transcendental, and expressed in terms of itself, just as we saw the derivatives of the trigonometric functions are expressed in terms of the trigonometric functions. That of the logarithm is algebraic, as those of the inverse trigonometric functions are. The exponential, like the trigonometric functions, is single-valued, while when imaginary values are taken into account, the logarithm is many-valued. Or, if we restrict ourselves to real values only, the exponential, like the trigonometric functions, assumes a definite value for any definite value of the variable, while the logarithm, like the inverse trigonometric functions (excepting $\tan^{-1} x$ and $\cot^{-1} x$), fails to have a value for certain values of the variable.

Note. — The method by which XXIV was obtained is another illustration of the application of the general method of Note 2 of the previous section. A very important further application is to the case where

$y = v^u$, u and v being functions of x. Then

$$\log y = u \log v, \text{ and}$$

$$\frac{y'}{y} = u' \log v + \frac{uv'}{v}; \text{ whence}$$

$$y' = v^u u' \log v + uv^{u-1}v'.$$

It is interesting to note that the derivative of v^u is the sum of the derivatives obtained by considering first v, then u separately constant, using XXIV and III′ respectively.

EXERCISES

Differentiate the following:

1. $y = e^{x^2}$.
2. $y = xe^x$.
3. $y = e^{\cos x}$
4. $y = \log (e^x + e^{-x})$.
5. $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.
6. $y = a^{\sin x}$.
7. $y = x^{\sin x}$.
8. $y = e^{ax} \sin bx$.
9. $y = x^2 e^{x^2}$.
10. $y = \tan^{-1} e^x$.

46. Summary. Combined with the formulas of differentiation given in Chapter II, those of this chapter conclude the list of all that are necessary for a working knowledge[1] of the subject. For the purpose of convenient reference this second set of formulas is appended, with the recommendation, however, that the student promptly memorize them so as to be able to use them with accuracy and facility.

X′. $D_x \sin u = \cos u \, D_x u$

XI′. $D_x \cos u = - \sin u \, D_x u$

XII′. $D_x \tan u = \sec^2 u \, D_x u$

XIII′. $D_x \cot u = - \csc^2 u \, D_x u$

XIV′. $D_x \sec u = \sec u \tan u \, D_x u$

XV′. $D_x \csc u = - \csc u \cot u \, D_x u$

XVI′. $D_x \sin^{-1} u = \dfrac{D_x u}{\sqrt{1 - u^2}}$

XVII′. $D_x \cos^{-1} u = \dfrac{- D_x u}{\sqrt{1 - u^2}}$

XVIII′. $D_x \tan^{-1} u = \dfrac{D_x u}{1 + u^2}$

XIX′. $D_x \cot^{-1} u = \dfrac{- D_x u}{1 + u^2}$

XX′. $D_x \sec^{-1} u = \dfrac{D_x u}{u\sqrt{u^2 - 1}}$

For some forms of u it is simpler to note that $\sec^{-1} u = \cos^{-1}\dfrac{1}{u}$, and use XVII′.

XXI′. $D_x \csc^{-1} u = \dfrac{- D_x u}{u\sqrt{u^2 - 1}}$.

For some forms of u it is simpler to note that $\csc^{-1} u = \sin^{-1}\dfrac{1}{u}$, and use XVI′.

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[1] The hyperbolic functions and their inverses will be treated separately in Part I of the Appendix of this book, so as not to break the continuity of the development of the subject. Their study may be taken up at any convenient time during the course.

XXII′. $$D_x \operatorname{vers}^{-1} u = \frac{D_x u}{\sqrt{2u - u^2}}.$$

XXIII′. $$D_x \log_a u = \frac{D_x u}{u} \log_a e.$$

XXIII″. $$D_x \log u = \frac{D_x u}{u}.$$

XXIV. $$D_x a^u = a^u D_x u \log a.$$

XXIV′. $$D_x e^u = e^u D_x u.$$

XXIV″. $$D_x e^x = e^x.$$

EXERCISES

Differentiate the following:

1. $y = \log \sqrt{\dfrac{1 + \sin x}{1 - \sin x}}.$

2. $y = \sin(\log x).$

3. $y = x \sec^{-1} \dfrac{1}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}.$

4. $y = \log \dfrac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a}.$

5. $y = \log \dfrac{ax}{b + \sqrt{x^2 + b^2}}.$

6. $y = \log \dfrac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} - x}.$

7. $y = -\dfrac{x}{\sqrt{x^2 + a^2}} + \log(x + \sqrt{x^2 + a^2}).$

8. $y = -\dfrac{\sqrt{x^2 + a^2}}{x} + \log(x + \sqrt{x^2 + a^2}).$

9. $y = x \cos^{-1} \dfrac{1}{x} - \log(x + \sqrt{x^2 - 1}).$

10. $y = \sin^{-1}\sqrt{\dfrac{x-a}{b-a}}.$

11. $y = \log \sin^{-1} x.$

12. $y = \log \cos^{-1}\dfrac{1}{x}.$

13. $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}.$

14. $y = \tan^{-1}\dfrac{2x}{1-x^2}.$

15. $y = \tan^{-1}(2\tan x).$

16. $y = \sin^{-1} e^x.$

17. $y = x^{\log x}.$

18. $y = x^{e^x}.$

19. $y = e^{e^x}.$

20. $y = e^{x^x}.$

21. $y = e^{\tan^{-1}x}.$

22. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}.$

23. $y = x^{\sin^{-1}x}.$

24. $y = \tan^{-1}\dfrac{\sqrt{x^2-a^2}}{a}.$

25. $y = \text{vers}^{-1}\, 2x^2.$

26. $y = \cos^{-1}\dfrac{1-x^2}{1+x^2}.$

27. $y = \dfrac{x^2\sqrt{(x+3)^3}}{\sqrt{x^2-1}}.$

28. $y = \log \tan^2 x + 2\cos 2x.$

29. $y = e^x(\sin x + \cos x).$

30. $y = \tan^{-1}\dfrac{e^x - e^{-x}}{2}.$

47. The *n*th derivative. Leibnitz's theorem for the *n*th derivative of a product. In general, the successive derivatives of a function become more and more complicated. There are some striking exceptions to this rule, the most important of which are enumerated below.

1. If $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2} + \cdots$. So that for n, any positive integer,

$$y^{(n)} \text{ or } D^n x^m = m(m-1)\cdots(m-n+1)x^{m-n}.$$

This holds for m, any real number. If m is a positive integer and $n > m$, $y^{(n)} = 0$, because then $y^{(m)} = m\,!$, a constant; so its successive derivatives are all zero.

2. If $y = \log x$, $y' = \dfrac{1}{x}$, $y'' = -\dfrac{1}{x^2}$, $y''' = \dfrac{2}{x^3}$, $y^{\text{iv}} = \dfrac{6}{x^4}$, $\cdots$. It is easy to show that

$$y^{(n)} \text{ or } D^n \log x = \frac{(-1)^{n+1}(n-1)\,!}{x^n}.$$

3. If $y = \sin ax$, $y' = a\cos ax$, $y'' = -a^2\sin ax$, $y''' = -a^3\cos ax$, $y^{\text{iv}} = a^4\sin ax$, $\cdots$. The general law is evidently

$$y^{(n)} \text{ or } D^n \sin ax = a^n \sin\left(ax + \frac{n\pi}{2}\right).$$

It is worth noting, and it is not difficult to remember, that *the even derivatives of the sine are expressed in terms of the sine, the algebraic sign being alternately negative and positive;* thus

$$D^{2n}\sin ax = (-1)^n a^{2n}\sin ax,$$

and *the odd derivatives of the sine are expressed in terms of the cosine, the algebraic sign being alternately positive and negative;* thus

$$D^{2n-1}\sin ax = (-1)^{n+1}a^{2n-1}\cos ax.$$

4. If $y = \cos ax$, $y' = -a\sin ax$, $y'' = -a^2\cos ax$, $y''' = a^3\sin ax$, $y^{\text{iv}} = a^4\cos ax$. The general law is

$$y^{(n)} \text{ or } D^n \cos ax = a^n \cos\left(ax + \frac{n\pi}{2}\right).$$

As before, it may be easier to remember and apply this if expressed thus: *The even derivatives of the cosine are expressed in terms of the cosine, the algebraic sign being alternately negative and positive;* thus

$$D^{2n}\cos ax = (-1)^n a^{2n}\cos ax,$$

and *the odd derivatives of the cosine are expressed in terms of the sine, the algebraic sign being alternately negative and positive;* thus

$$D^{2n-1}\cos ax = (-1)^n a^{2n-1}\sin ax.$$

5. If $y = e^{ax}$, $y' = ae^{ax}$, $y'' = a^2e^{ax}$, $\cdots$. Evidently

$$y^{(n)} \text{ or } D^n e^{ax} = a^n e^{ax}.$$

6. If $y = uv$, where u and v are functions of x, we have (VIII, § 12),

$$\begin{aligned}
y' &= u'v + uv'.\\
y'' &= u''v + 2\,u'v' + uv''.\\
y''' &= u'''v + 3\,u''v' + 3\,u'v'' + uv'''.\\
y^{\text{iv}} &= u^{\text{iv}}v + 4\,u'''v' + 6\,u''v'' + 4\,u'v''' + uv^{\text{iv}}.
\end{aligned}$$

. . . .

This reminds one strikingly of the Binomial Theorem Thus, if we write, for example,

$$(a+b)^4 = a^4b^0 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + a^0b^4,$$

as we may, since $a^0 = 1$ and $b^0 = 1$, and if we agree to consider the zero-th derivative of a function as identical with the function, we can write in symbolic form

$$D_x^2(uv) = (D_xu + D_xv)^2.$$
$$D_x^3(uv) = (D_xu + D_xv)^3.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

By mathematical induction we can establish the general formula

$$\begin{aligned} D_x^n(uv) &= (D_xu + D_xv)^n \\ &= u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2\,!}u^{(n-2)}v'' + \cdots \\ &\qquad + nu'v^{(n-1)} + uv^{(n)}. \end{aligned}$$

This formula, known as *Leibnitz's theorem* for the nth derivative of a product of two functions, seems worth noting here because it is a good check in case one actually carries out the successive derivatives of the product, and also because it enables one to write down readily the nth derivative of the product without calculating the earlier derivatives, especially if u and v are functions of the kind considered in the earlier part of this section.

EXERCISES

1. Find the fourth derivative of $e^{ax}\cos bx$.

Since

$$D^ne^{ax} = a^ne^{ax},\quad D\cos bx = -b\sin bx,\quad D^2\cos bx = -b^2\cos bx,$$
$$D^3\cos bx = b^3\sin bx,\qquad D^4\cos bx = b^4\cos bx,$$

we have, using Leibnitz's theorem,

$$\begin{aligned} D^4e^{ax}\cos bx &= a^4e^{ax}\cos bx - 4\,a^3be^{ax}\sin bx - 6\,a^2b^2e^{ax}\cos bx \\ &\qquad + 4\,ab^3e^{ax}\sin bx + b^4e^{ax}\cos bx \\ &= (a^4 - 6\,a^2b^2 + b^4)e^{ax}\cos bx - 4\,ab(a^2 - b^2)e^{ax}\sin bx. \end{aligned}$$

The student may check this by actually differentiating the function four times.

2. Find the fifth derivative of $x^3 \sin 2x$.

3. Find the sixth derivative of x^4e^{2x}.

4. Find the fourth derivative of $\sin x \log x^2$.

5. If n is a positive integer, show that $D^{n+1}x^n \log x = \frac{n!}{x}$.

6. If $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, show that $y'' = \frac{y}{a^2}$.

7. Writing $\frac{1-x}{1+x}$ as $\frac{2-1-x}{1+x}$ or $\frac{2}{1+x} - 1$, find its nth derivative.

8. Find the nth derivative of a^x.

9. Writing $\frac{x^3}{1+x}$ as $\frac{x^3+1-1}{1+x} = x^2 - x + 1 - \frac{1}{1+x}$, find its second derivative; also its nth derivative.

10. If $y = xe^{-x}$, show that $D^3y - D^2y - 2\,Dy = 3\,e^{-x}$.

11. If $y = \cos x \cdot \log \cos x + x \sin x$, show that $D^2y + y = \sec x$.

12. If $y = (\log x)^2 + 2 \log x - 3$, show that
$$x^4D^4y + 6\,x^3D^3y + 9\,x^2D^2y + 3\,xDy + y = (1 + \log x)^2.$$

13. If $y = x \sin 2x$, show that $D^2y + 4\,y = 4 - 8 \sin^2 x$.

14. If $y = \sin x \log(\tan x + \sec x)$, show that $D^2y + y$
$$= \sec^2 x + 1.$$

CHAPTER VII

METHODS OF INTEGRATION

48. Formulas of integration. Since $\int f(x)\,dx = F(x)$, provided $D_xF(x) = f(x)$ or $dF(x) = f(x)\,dx$, each formula of differentiation should be examined with the view of finding a corresponding formula of integration. In § 36, we noted that corresponding to

III′. $du^n = nu^{n-1}\,du$ we have

A. $$\int u^m\,du = \frac{u^{m+1}}{m+1} + c. \qquad m \neq -1.$$

From the last part of Note 2, § 44, we see now that this formula holds for m any real number except -1. When $m = -1$, we have from

XXIII‴. $d \log u = \dfrac{du}{u}$,

B. $$\int \frac{du}{u} = \log u + c.$$

Remark. — Our integration formulas are given in terms of the variable u, to emphasize that their form does not depend upon the variable appearing in them. Frequently a change of variable (§ 36) reduces a given integral to one appearing in our list of integration formulas. The following exercises will illustrate.

Ex. 1. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1}}$. Since $x\,dx$ is the differential of $x^2 + 1$, to within the numerical factor 2, the transformation $x^2 + 1 = u$ is suggested. Making this, we have

$$\int \frac{x\,dx}{\sqrt{x^2+1}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + c = \sqrt{x^2+1} + c.$$

Ex. 2. $\int \tan x\, dx$. Writing this in the form $\int \frac{\sin x\, dx}{\cos x}$, and noting that the numerator is the differential of the denominator, to within the algebraic sign, we let $\cos x = u$. Differentiating, we have $\sin x\, dx = - du$; whence

$$\int \tan x\, dx = -\int \frac{du}{u} = - \log u + c = \log \frac{1}{u} + c$$
$$= - \log \cos x + c = \log \sec u + c.$$

Ex. 3. $\int \frac{(x-1)^2\, dx}{x^2}$. Writing $\frac{(x-1)^2}{x^2} = \frac{x^2 - 2x + 1}{x^2} = 1 - \frac{2}{x} + \frac{1}{x^2}$,

$$\int \frac{(x-1)^2\, dx}{x^2} = x - \log x^2 - \frac{1}{x} + c.$$

Ex. 4. $\int \frac{x^3\, dx}{x-1}$. Writing $\frac{x^3}{x-1} = \frac{x^3 - 1 + 1}{x - 1} = x^2 + x + 1 + \frac{1}{x-1}$,

$$\int \frac{x^3\, dx}{x-1} = \frac{x^3}{3} + \frac{x^2}{2} + x + \log (x-1) + c.$$

Ex. 5. $\int \frac{dx}{ax+b}$. Letting $ax + b = u$, $dx = \frac{du}{a}$. Hence

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a} \log u + c = \frac{1}{a} \log (ax+b) + c = \log (ax+b)^{\frac{1}{a}} + c.$$

EXERCISES

Evaluate the following:

1. $\int \frac{(3x^3 - 4x + 2)}{x}\, dx.$ 2. $\int \frac{x^3\, dx}{\sqrt{x^4 - 3}}.$

3. $\int \frac{x^3\, dx}{x^4 - 3}.$ 7. $\int (x-1)^{\frac{3}{4}}\, dx.$ 11. $\int \frac{(3x+1)\, dx}{3x^2 + 2x - 1}.$

4. $\int x^2 \sqrt{1 - x^3}\, dx.$ 8. $\int \frac{1+x}{1-x}\, dx.$ 12. $\int \frac{e^{ax} + e^{-ax}}{e^{ax} - e^{-ax}}\, dx.$

5. $\int \sqrt{ax + b}\, dx.$ 9. $\int (1-x)^5\, dx.$ 13. $\int \tan x \sec^2 x\, dx.$

6. $\int \cos x \csc^3 x\, dx.$ 10. $\int \frac{a \sin x\, dx}{b + c \cos x}.$ 14. $\int \frac{\tan^{-1} x}{1 + x^2}\, dx.$

15. $\int \frac{\log x\, dx}{x}.$ 16. $\int \frac{e^x\, dx}{1 + e^x}.$

17. $\int \sin^3 x\, dx = \int \sin x (1 - \cos^2 x)\, dx.$

18. $\int \frac{dx}{(1 + x^2) \tan^{-1} x}.$ 19. $\int \frac{dx}{x\sqrt{\log x + 1}}.$ 20. $\int \cot x\, dx.$

49. Additional formulas of integration. Just as the formulas of integration **A** and **B** followed at once from the corresponding formulas of differentiation, so we obtain the following:

From

X′. $d \sin u = \cos u\, du.$ **C.** $\int \cos u\, du = \sin u + c.$

XI′. $d \cos u = - \sin u\, du.$ **D.** $\int \sin u\, du = - \cos u + c.$

XII′. $d \tan u = \sec^2 u\, du.$ **E.** $\int \sec^2 u\, du = \tan u + c.$

XIII′. $d \cot u = - \csc^2 u\, du.$ **F.** $\int \csc^2 u\, du = - \cot u + c.$

XIV′. $d \sec u = \sec u \tan u\, du.$ **G.** $\int \sec u \tan u\, du = \sec u + c.$

XV′. $d \csc u = - \csc u \cot u\, du.$ **H.** $\int \csc u \cot u\, du = - \csc u + c.$

$$\left.\begin{array}{ll} \text{XVI}'. & d \sin^{-1} u = \dfrac{du}{\sqrt{1-u^2}}. \\ \text{XVII}'. & d \cos^{-1} u = \dfrac{du}{-\sqrt{1-u^2}}. \end{array}\right\} \quad \textbf{J.}^1 \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = -\cos^{-1} u + c.$$

$$\left.\begin{array}{ll} \text{XVIII}'. & d \tan^{-1} = \dfrac{du}{1+u^2}. \\ \text{XIX}'. & d \cot^{-1} u = \dfrac{-du}{1+u^2}. \end{array}\right\} \quad \textbf{K.} \int \frac{du}{1+u^2} = \tan^{-1} u + c = -\cot^{-1} u + c.$$

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[1] Since $\sin^{-1} u + \cos^{-1} u = \frac{\pi}{2}$, *i.e.* $\sin^{-1} u = - \cos^{-1} u + \frac{\pi}{2}$, and $\tan^{-1} u = - \cot^{-1} u + \frac{\pi}{2}$, and $\sec^{-1} u = - \csc^{-1} u + \frac{\pi}{2}$, for $u > 0$ and $\sec^{-1} u = - \csc^{-1} u - \frac{3\pi}{2}$ for $u > 0$, the two values of each of the integrals **J**, **K**, and **L** are readily accounted for.

$$\left.\begin{array}{ll}\text{XX}'. & d\sec^{-1}u = \dfrac{du}{u\sqrt{u^2-1}}. \\ \text{XXI}'. & d\csc^{-1}u = \dfrac{du}{-u\sqrt{u^2-1}}.\end{array}\right\} \quad \textbf{L}.\ \int\frac{du}{u\sqrt{u^2-1}} \begin{array}{l} \\ = \sec^{-1}u + c \\ = -\csc^{-1}u + c.\end{array}$$

$$\text{XXII}'.\ d\,\text{vers}^{-1}u = \frac{du}{\sqrt{2u-u^2}}. \qquad \textbf{M}.\ \int\frac{du}{\sqrt{2u-u^2}} \begin{array}{l} \\ = \text{vers}^{-1}u + c \\ = \cos^{-1}(1-u).\end{array}$$

$$\text{XXIV}'.\ de^u = e^u\,du. \qquad \textbf{N}.\ \int e^u\,du = e^u + c.$$

$$\text{XXIV}.\ da^u = a^u \log a\,du. \qquad \textbf{N}'.\ \int a^u\,du = \frac{a^u}{\log a} + c.$$

The alternate value in **M** comes from the following consideration. Noting that $2u - u^2 = 1 - 1 + 2u - u^2 = 1 - (1-u)^2$

$$\int\frac{du}{\sqrt{2u-u^2}} = \int\frac{du}{\sqrt{1-(1-u)^2}} = -\int\frac{d(1-u)}{\sqrt{1-(1-u)^2}}$$
$$= \cos^{-1}(1-u) + c.$$

That these two values of the integral are consistent follows from the definition of $\text{vers}\,y = 1 - \cos y$. For if we write $y = \text{vers}^{-1}u$, then $u = \text{vers}\,y = 1 - \cos y$. $\therefore \cos y = 1 - u$ and $y = \cos^{-1}(1-u) = \text{vers}^{-1}u$.

If in $\displaystyle\int\frac{du}{u\sqrt{u^2-1}}$ we put $u = \frac{1}{t}$, then $du = -\frac{dt}{t^2}$ and

$$\int\frac{du}{u\sqrt{u^2-1}} = -\int\frac{dt}{\sqrt{1-t^2}} = \cos^{-1}t + c = \cos^{-1}\frac{1}{u} + c.$$

This additional value of the integral in **L** is worth emphasizing.

The following modifications of formulas **J**, **K**, **L**, **M** will be found worth while.

Noting that $\int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{a}\int \frac{du}{\sqrt{1-\left(\frac{u}{a}\right)^2}}$, let $\frac{u}{a}=t$. The integral reduces to

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} + c = \sin^{-1}\frac{u}{a} + c \quad \text{or} \quad -\cos^{-1}\frac{u}{a} + c.$$

Hence **J** may be replaced by the more general form

$$\textbf{J}'. \quad \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c \quad \text{or} \quad -\cos^{-1}\frac{u}{a} + c.$$

Noting that $\int \frac{du}{a^2+u^2} = \frac{1}{a^2}\int \frac{du}{1+\left(\frac{u}{a}\right)^2}$, let $\frac{u}{a}=t$. The integral reduces to

$$\frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\tan^{-1}t + c = \frac{1}{a}\tan^{-1}\frac{u}{a} + c \quad \text{or} \quad -\frac{1}{a}\cot^{-1}\frac{u}{a} + c.$$

Hence **K** may be replaced by the more general form

$$\textbf{K}'. \quad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c \quad \text{or} \quad -\frac{1}{a}\cot^{-1}\frac{u}{a} + c.$$

Noting that $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\int \frac{du}{u\sqrt{\left(\frac{u}{a}\right)^2-1}}$, let $\frac{u}{a}=t$. The integral reduces to

$$\frac{1}{a}\int \frac{dt}{t\sqrt{t^2-1}} = \frac{1}{a}\sec^{-1}t + c = \frac{1}{a}\sec^{-1}\frac{u}{a} + c \quad \text{or} \quad -\frac{1}{a}\csc^{-1}\frac{u}{a} + c.$$

Hence **L** may be replaced by the more general form

$$\textbf{L}'. \quad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c \quad \text{or} \quad -\frac{1}{a}\csc^{-1}\frac{u}{a} + c$$

$$= \frac{1}{a}\cos^{-1}\frac{a}{u} + c \quad \text{or} \quad -\frac{1}{a}\sin^{-1}\frac{a}{u} + c.$$

Noting that $\int\frac{du}{\sqrt{2\,au-u^2}}=\frac{1}{a}\int\frac{du}{\sqrt{\frac{2\,u}{a}-\left(\frac{u}{a}\right)^2}}$, let $\frac{u}{a}=t$. The integral reduces to

$$\int\frac{dt}{\sqrt{2\,t-t^2}}=\text{vers}^{-1}t+c=\text{vers}^{-1}\frac{u}{a}+c \text{ or } \cos^{-1}(1-t)+c$$
$$=\cos^{-1}\frac{a}{a-u}+c.$$

Hence **M** may be replaced by the more general form

$$\textbf{M}'.\quad \int\frac{du}{\sqrt{2\,au-u^2}}=\text{vers}^{-1}\frac{u}{a}+c \text{ or } \cos^{-1}\frac{a-u}{a}+c.$$

Remark. — At times, students find it confusing to remember when the extra factor $\frac{1}{a}$ is to be used in these integrals and when it is to be omitted. It is needless to emphasize that such confusion is highly undesirable. It may be sufficient to bear in mind that the factor $\frac{1}{a}$ does *not* occur in the cases where the values of the integrals involve $\sin^{-1}\frac{u}{a}$, $\cos^{-1}\frac{u}{a}$, and $\text{vers}^{-1}\frac{u}{a}$, and *does* occur with the functions $\tan^{-1}\frac{u}{a}$, $\cot^{-1}\frac{u}{a}$, $\sec^{-1}\frac{u}{a}$, and $\csc^{-1}\frac{u}{a}$.

To those who prefer a more logical way of remembering the rule, the following is suggested by the transformation $\frac{u}{a}=t$ in the deduction of the formulas. Giving the weight 1 to u, du, and the square root of a quadratic function of u, and the weight 2 to a product of any two of these or to a quadratic function of u, the rule is that *the factor* $\frac{1}{a}$ *does not appear when the numerator and denominator have the same weight, but it does appear when the weight of the denominator exceeds that of the numerator by 1.*

The following exercises will illustrate the use of the formulas of this section, and also some of the more common devices usually employed in reducing the given integrals to known forms:

Ex. 1. $\int x \sin x^2\, dx.$ Since $d(x^2) = 2\,x\,dx$, put $x^2 = u$; then $x\,dx = \frac{1}{2}\,du$, and

$$\int x \sin x^2\, dx = \tfrac{1}{2}\int \sin u\, du = -\tfrac{1}{2}\cos u + c = -\tfrac{1}{2}\cos x^2 + c.$$

Ex. 2. $\int e^{\tan x}\sec^2 x\, dx.$ Since $d\tan x = \sec^2 x\, dx$, put $\tan x = u$; then

$$\int e^{\tan x}\sec^2 x\, dx = \int e^u\, du = e^u + c = e^{\tan x} + c.$$

Ex. 3. $\int \frac{dx}{x^2 - 2\,x + 4}\cdot$ Noting that

$$x^2 - 2\,x + 4 = x^2 - 2\,x + 1 + 3 = (x-1)^2 + (\sqrt{3})^2,$$

$$\int \frac{dx}{x^2 - 2\,x + 4} = \int \frac{du}{u^2 + (\sqrt{3})^2},$$ where $u = x - 1$. Using **K′**

$$\int \frac{dx}{x^2 - 2\,x + 4} = \int \frac{du}{u^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}} + c.$$

Ex. 4. $\int \frac{dx}{\sqrt{2 - 4\,x - x^2}}\cdot$ Adding and subtracting 4 to the quadratic expression under the radical, it takes the form of the difference of two squares, thus

$$2 - 4\,x - x^2 = 6 - 4 - 4\,x - x^2 = (\sqrt{6})^2 - (x+2)^2;$$

and the integral takes the form

$$\int \frac{du}{\sqrt{(\sqrt{6})^2 - u^2}},$$ where $u = x + 2$.

Using **J′**, we have

$$\int \frac{dx}{\sqrt{2 - 4\,x - x^2}} = \int \frac{du}{\sqrt{(\sqrt{6})^2 - u^2}} = \sin^{-1}\frac{u}{\sqrt{6}} + c$$
$$= \sin^{-1}\frac{x+1}{\sqrt{6}} + c.$$

Ex. 5. $\int \frac{2x-5}{2x^2-2x+5}dx$. Since $d(2x^2-2x+5)=(4x-2)dx$ $=2(2x-1)dx$, if we write $2x-5=2x-1-4$, our integral may be written as the algebraic sum[1]

$$\int \frac{2x-1}{2x^2-2x+5}dx - 4\int \frac{dx}{2x^2-2x+5}.$$

To evaluate the first of these, let $2x^2-2x+5=u$; then $(2x-1)dx=\frac{1}{2}du$, and

$$\int \frac{2x-1}{2x^2-2x+5}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log u + c_1 = \log\sqrt{u} + c_1$$

$$= \log\sqrt{2x^2-2x+5} + c_1.$$

To evaluate the second integral, reduce it to the form appearing in **K′**; that is, write it

$$-4\int \frac{dx}{2(x^2-x+\frac{5}{2})} = -2\int \frac{dx}{x^2-x+\frac{1}{4}+\frac{9}{4}}$$

$$= -2\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{3}{2})^2} = -2\int \frac{du}{u^2+(\frac{3}{2})^2},$$

where $u=x-\frac{1}{2}$. Using **K′**, this has the value

$-2\cdot\frac{2}{3}\tan^{-1}\frac{2u}{3}+c_2 = -\frac{4}{3}\tan^{-1}\frac{2x-1}{3}+c_2$. Hence

$$\int \frac{2x-1}{2x^2-2x+5}dx = \log\sqrt{2x^2-2x+5} - \frac{4}{3}\tan^{-1}\frac{2x-1}{3} + c,$$

where the single constant replaces the sum c_1+c_2.

---

[1] To write the integral as the sum $2\int \frac{x\,dx}{2x^2-2x+5} - 5\int \frac{dx}{2x^2-2x+5}$ and proceed to evaluate each would lead to a distinct loss of time and effort, since the problem of evaluating $2\int \frac{x\,dx}{2x^2-2x+5}$ alone requires the evaluation of $\int \frac{2x-1}{2x^2-2x+5}dx + \int \frac{dx}{2x^2-2x+5}$.

Ex. 6. $\int \frac{dx}{e^x + e^{-x}}$. Multiply numerator and denominator by e^x. Then, noting that $de^x = e^x dx$,

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1} = \tan^{-1} e^x + c.$$

Ex. 7. $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$. Multiplying numerator and denominator by e^{-x}, we have

$$\int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log (e^x + e^{-x}) + c.$$

Ex. 8. $\int \frac{dx}{\sqrt{x - x^2}}$. Writing this in the form $\int \frac{dx}{\sqrt{x}\sqrt{1 - x}}$ and noting that $d\sqrt{x} = \frac{dx}{2\sqrt{x}}$, the introduction of the new variable $\sqrt{x} = u$ leads to

$$\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{dx}{\sqrt{x}\sqrt{1 - x}} = 2\int \frac{du}{\sqrt{1 - u^2}} = 2 \sin^{-1} u + c$$
$$= 2 \sin^{-1} \sqrt{x} + c.$$

As an exercise, the student should evaluate this integral by the method of Ex. 4. Doing this, he will find $\int \frac{dx}{\sqrt{x - x^2}} = \sin^{-1} (2x - 1) + c$. He should then show that the two results are consistent.

Ex. 9. $\int \cos^2 x\, dx$. Noting that $\cos^2 x = \frac{1 + \cos 2x}{2}$, we have $\int \cos^2 x\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + c.$

Ex. 10. $\int \cos^3 x\, dx$. Noting that $\cos^2 x = 1 - \sin^2 x$,

$$\int \cos^3 x\, dx = \int \cos x (1 - \sin^2 x)\, dx = \int \cos x\, dx - \sin^2 x \cos x\, dx$$
$$= \sin x - \tfrac{1}{3} \sin^3 x + c.$$

NOTE. — The devices used in the illustrative exercises to reduce given integrals to the forms in which they appear in the integration formulas are worth noting. In the main they consist in the following:

(*a*) The introduction of a new variable, suggested by the form of the integrand. [In the integral $\int f(x)\,dx$, the function $f(x)$ is known as the *integrand.*]

(*b*) The breaking up of the integrand into a sum of terms, each of which comes under a known head (Exs. 5, 9, 10).

(*c*) The addition and subtraction of a constant in a quadratic expression in the variable, causing this expression to take the form of the sum or difference of two squares (Exs. 3, 4, 5).

(*d*) The introduction of a common variable factor in numerator or denominator of the integrand so as to bring about a marked simplification (Exs. 6, 7).

(*e*) Other possible simplifications through algebraic and trigonometric manipulations (Exs. 8, 9, 10 and Exercise 10 below).

It cannot be emphasized too much that definite acquaintance with *all* the important formulas of integration, so that there is no doubt in the student's mind concerning them, is a prerequisite for any success in integration.

EXERCISES

Evaluate the following:

1. $\int \sin 2x\,dx.$

2. $\int x\cos x^2\,dx.$

3. $\int \frac{dx}{\sqrt{x}(x+1)}.$

4. $\int xe^{-x^2}\,dx.$

5. $\int \sin^2 x\,dx.$

6. $\int \frac{x\,dx}{\sqrt{5+4x-x^2}}.$

7. $\int e^{\sin x}\cos x\,dx.$

8. $\int \frac{dx}{\sqrt{e^{2x}-1}}.$

9. $\int \tan^2 x\,dx.$

10. $\int \sqrt{\frac{1-x}{1+x}}\,dx.$

HINT. — In Ex. 10, rationalize the numerator.

11. $\int \frac{1-x}{1+x^2}\,dx.$

12. $\int \frac{dx}{x\sqrt{x-1}}.$

13. $\int \sin^3 x\,dx.$

14. $\int \frac{dx}{x^2+6x+13}.$

15. $\int \frac{\sec^2\sqrt{x}\,dx}{\sqrt{x}}.$

16. $\int x\csc^2 x^2\,dx.$

17. $\int \sin^2 x\tan x\,dx.$

18. $\int \sin x\sec^4 x\,dx.$

19. $\int \frac{dx}{x+x\log^2 x}.$

20. $\int \frac{x^2\,dx}{\sqrt{a^6-x^6}}.$

50. Other formulas of integration. The formulas given in the two previous sections were suggested by the corresponding formulas of differentiation. Some others, analogous to these, but not obtainable as they were, will be added here.

Consider $\int \frac{du}{a^2 - u^2}$. Noting that

$$\frac{1}{a^2 - u^2} = \frac{1}{2a}\left(\frac{1}{a+u} + \frac{1}{a-u}\right),$$

we can write

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\int \frac{du}{a+u} + \frac{1}{2a}\int \frac{du}{a-u}$$

$$= \frac{1}{2a}\log(a-u) + \frac{1}{2a}\log(a-u) + c = \frac{1}{2a}\log\frac{a+u}{a-u} + c. \qquad (1)$$

Since the logarithm of a negative number is imaginary, it is desirable to find another form for this integral in case $u > a$. To do this note that, in an analogous manner,

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int \frac{du}{u-a} - \frac{1}{2a}\int \frac{du}{u+a} = \frac{1}{2a}\log\frac{u-a}{u+a} + c. \qquad (2)$$

$$\therefore \int \frac{du}{a^2 - u^2} = -\frac{1}{2a}\log\frac{u-a}{u+a} + c^{1} = \frac{1}{2a}\log\frac{u+a}{u-a} + c. \qquad (3)$$

Similarly, taking the negatives of both sides of (1), we have the companion formula

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{a-u}{a+u} + c, \qquad (4)$$

which gives a real form for the integral when $u < a$. We have thus arrived at the formulas

O. $$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\log\frac{a+u}{a-u} + c \quad \text{or} \quad \frac{1}{2a}\log\frac{u+a}{u-a} + c.$$

O′. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{a-u}{a+u} + c \quad \text{or} \quad \frac{1}{2a}\log\frac{u-a}{u+a} + c,$$

---

[1] There is no need of changing the sign of the constant of integration, since an arbitrary constant may be either positive or negative.

where the companion forms are given to be used when $u < a$ and $u > a$, respectively.

It will be noted that in both forms for **O** the sum appears in the numerator and the difference in the denominator; in the case of **O′** the difference appears in the numerator and the sum in the denominator.

Consider $\int \frac{du}{\sqrt{u^2+k}}$, where k may be positive or negative. A perfectly general method, which we shall consider later (§ 54), suggests the new variable t defined by

$\sqrt{u^2+k} = t - u$. Squaring both sides, we have

$$k = t^2 - 2\,tu. \quad \text{Differentiating}$$

$$0 = 2(t-u)\,dt - 2\,tdu. \quad \therefore du = \frac{t-u}{t}\,dt = \frac{\sqrt{u^2+k}}{t}\,dt.$$

$$\therefore \int \frac{du}{\sqrt{u^2+k}} = \int \frac{dt}{t} = \log t + c = \log\left(u + \sqrt{u^2+k}\right) + c.$$

We have thus the important formula

P.
$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + c,$$

where we have replaced k by $\pm a^2$ to emphasize that it is a positive or negative constant.

The appearance of the multiplier $\frac{1}{a}$ in **O** and **O′** and its absence in **P** are consistent with the rule given in the Remark, § 49.

In the second illustrative exercise of § 48 we proved

Q.
$$\int \tan u\,du = -\log \cos u + c = \log \sec u + c.$$

In an analogous manner the student should prove that

R.
$$\int \cot u\,du = \log \sin u + c.$$

Consider now $\int \sec u\, du$. Let $\sec u = t$, then $\sec u \tan u\, du = dt$. $\therefore\ \sec u\, du = \frac{dt}{\tan u} = \frac{dt}{\sqrt{t^2-1}}$. Hence, using **P**,

$$\int \sec u\, du = \int \frac{dt}{\sqrt{t^2-1}} = \log(t+\sqrt{t^2-1})+c$$
$$= \log(\sec u + \tan u) + c.$$

This formula should be added to our list.

S. $$\int \sec u\, du = \log(\sec u + \tan u) + c.$$

To evaluate $\int \csc u\, du$, we can let $\csc u = t$ and proceed as before, or we can write $\csc u = \sec\left(\frac{\pi}{2} - u\right)$, let $\frac{\pi}{2} - u = t$, and use **S**. By either method we find

T. $$\int \csc u\, du = -\log(\csc u + \cot u) + c = \log(\csc u - \cot u) + c.$$

The second form follows from the fact that

$$\frac{1}{\csc u + \cot u} = \csc u - \cot u.$$

Ex. 1. $\int \frac{3x+2}{4x^2-4x-5}\, dx$. Noting that

$$4x^2 - 4x - 5 = 4x^2 - 4x + 1 - 6 = (2x-1)^2 - 6,$$

let $\qquad 2x - 1 = u \qquad$ Then $x = \frac{u+1}{2}$, $dx = \frac{1}{2}du$,

and the integral takes the form[1]

$$\frac{1}{4}\int \frac{3u+7}{u^2-6}\, du = \frac{3}{4}\int \frac{u\, du}{u^2-6} + \frac{7}{4}\int \frac{du}{u^2-6}.$$

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[1] The student should compare this method of expressing the given integral as the sum of two auxiliary integrals with that employed in Ex. 5, § 49. While not essentially different, it is somewhat less direct; but it may be found simpler when fractional multiples of the auxiliary integrals are to be taken, as in this case.

Using **B**, $\frac{3}{4}\int\frac{u\,du}{u^2-6}=\frac{3}{8}\log(u^2-6)+c_1$

$$=\tfrac{3}{8}\log(4x^2-4x-5)+c_2.$$

Using **O′**, $\frac{7}{4}\int\frac{du}{u^2-6}=\frac{7}{8\sqrt{6}}\log\frac{u-\sqrt{6}}{u+\sqrt{6}}+c_2$

$$=\frac{7}{8\sqrt{6}}\log\frac{2x-1-\sqrt{6}}{2x-1+\sqrt{6}}+c_2.$$

$$\therefore\int\frac{3x+2}{4x^2-4x-5}dx=\tfrac{3}{8}\log(4x^2-4x-5)+\frac{7}{8\sqrt{6}}\log\frac{2x-1-\sqrt{6}}{2x-1+\sqrt{6}}+c.$$

Ex. 2. $\int\frac{5x\,dx}{\sqrt{2x^2-12x+23}}$. As before, noting that

$2x^2-12x+23=2(x-3)^2+5$, let $x-3=u$.

The integral then takes the form

$$\int\frac{5u+15}{\sqrt{2u^2+5}}du=5\int\frac{u\,du}{\sqrt{2u^2+5}}+15\int\frac{du}{\sqrt{2u^2+5}}.$$

Letting $2u^2+5=t$, and using **A**,

$$5\int\frac{u\,du}{\sqrt{2u^2+5}}=\frac{5}{4}\int\frac{dt}{\sqrt{t}}=\tfrac{5}{2}\sqrt{t}+c_2=\tfrac{5}{2}\sqrt{2u^2+5}+c.$$

For the second integral, we have, using **P**,

$$15\int\frac{du}{\sqrt{2u^2+5}}=\frac{15}{\sqrt{2}}\int\frac{du}{\sqrt{u^2+\frac{5}{2}}}=\frac{15}{\sqrt{2}}\log\left(u+\sqrt{u^2+\tfrac{5}{2}}\right)+c_2.$$

$$\therefore\int\frac{5x\,dx}{\sqrt{2x^2-12x+23}}$$

$$=\tfrac{5}{2}\sqrt{2x^2-12x+23}+\frac{15}{\sqrt{2}}\log\left[(x-3)+\sqrt{x^2-6x+\tfrac{23}{2}}\right]+c.$$

Ex. 3. $\int \frac{\cos x\,dx}{3-4\sin^2 x}$. Let $\sin x = u$. The integral reduces to $\int \frac{du}{3-4u^2} = \frac{1}{4}\int \frac{du}{\frac{3}{4}-u^2}$. Using **0**, we have

$$\int \frac{du}{\frac{3}{4}-u^2} = \frac{1}{\sqrt{3}} \log \frac{\frac{\sqrt{3}}{2}+u}{\frac{\sqrt{3}}{2}-u} + c = \frac{1}{\sqrt{3}} \log \frac{\sqrt{3}+2u}{\sqrt{3}-2u} + c.$$

$$\therefore \int \frac{\cos x\,dx}{3-4\sin^2 x} = \frac{1}{4\sqrt{3}} \log \frac{\sqrt{3}+2\sin x}{\sqrt{3}-2\sin x} + c.$$

EXERCISES

Evaluate the following:

1. $\int \frac{5x-3}{\sqrt{x^2-8x+23}}\,dx.$
2. $\int \sqrt{\frac{x-1}{x+1}}\,dx.$
3. $\int \cot x \sec x\,dx.$
4. $\int \frac{x\,dx}{1-x^4}.$
5. $\int \frac{2x-5}{3x^2-2}.$
6. $\int \frac{1}{x^2} \cot \frac{1}{x}\,dx.$
7. $\int \frac{4x+2}{4x^2-4x+5}.$
8. $\int \frac{x\,dx}{\sqrt{x^2+2x-5}}.$
9. $\int \frac{\sin x\,dx}{3-\cos^2 x}.$
10. $\int \frac{dx}{\sqrt{3-2x-x^2}}.$
11. $\int \frac{dx}{x^{\frac{2}{3}}(x^{\frac{2}{3}}+1)}.$
12. $\int \cot x \log \sin x\,dx.$

51. Standard forms. The formulas of integration deduced in the previous sections will be referred to as *standard forms.* It is very desirable that the student know them so as to be able to use them with accuracy and facility. To aid in this, they are given below, not in the order in which they were deduced, but in a more systematic arrangement according to integrands.

A. $$\int u^m\,du = \frac{u^{m+1}}{m+1} + c. \quad m \neq 1.$$

B. $$\int \frac{du}{u} = \log u + c.$$

K′. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c.$$

O. $$\int \frac{du}{a^2 - u^2} = \frac{1}{2\,a}\log\frac{a+u}{a-u} + c \text{ or } \frac{1}{2\,a}\log\frac{u+a}{u-a} + c.$$

O′. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2\,a}\log\frac{a-u}{a+u} + c \text{ or } \frac{1}{2\,a}\log\frac{u-a}{u+a} + c.$$

J′. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c \text{ or } -\cos^{-1}\frac{u}{a} + c.$$

P. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + c.$$

L′. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c \text{ or } -\frac{1}{a}\csc^{-1}\frac{u}{a} + c$$
$$= \frac{1}{a}\cos^{-1}\frac{a}{u} + c \text{ or } -\frac{1}{a}\sin^{-1}\frac{a}{u} + c.$$

M′. $$\int \frac{du}{\sqrt{2\,au - u^2}} = \operatorname{vers}^{-1}\frac{u}{a} + c = \cos^{-1}\frac{a-u}{a} + c.$$

N. $$\int e^u\,du = e^u + c.$$

N′. $$\int a^u\,du = \frac{a^u}{\log a} + c.$$

D. $$\int \sin u\,du = -\cos u + c.$$

C. $$\int \cos u\,du = \sin u + c.$$

Q. $$\int \tan u\,du = -\log\cos u + c = \log\sec u + c.$$

R. $$\int \cot u\, du = \log \sin u + c.$$

S. $$\int \sec u\, du = \log(\sec u + \tan u) + c$$
$$= \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + c. \quad \text{[See Ex. 1, § 57.]}$$

T. $$\int \csc u\, du = -\log(\csc u + \cot u) + c$$
$$= \log(\csc u - \cot u) + c = \log\tan\frac{u}{2} + c.$$

[See Ex. 1, § 57.]

E. $$\int \sec^2 u\, du = \tan u + c.$$

F. $$\int \csc^2 u\, du = -\cot u + c.$$

G. $$\int \sec u \tan u\, du = \sec u + c.$$

H. $$\int \csc u \cot u\, du = -\csc u + c.$$

All the remaining methods of integration consist in so manipulating the integrand as to bring it under one or another of the standard forms. In the succeeding sections of this chapter some of the more important of these methods will be developed.

52. Integration by parts. Corresponding to the formula for differentiation of a product

VIII. $$d(uv) = v\,du + u\,dv,$$

we have, upon integrating both members of the equation,

$$uv = \int v\,du + \int u\,dv,$$

or

U. $$\int u\,dv = uv - \int v\,du.$$

This most useful formula, usually referred to as that of *integration by parts*, enables one to replace the problem of evaluating an integral of the form $\int u\,dv$ by that of evaluating one of the form $\int v\,du$, which may be much more manageable. We shall have frequent occasion to use it. The following exercises will illustrate.

Ex. 1. $\int \log x\,dx$. Let $u = \log x$ and $dv = dx$.

Then $du = \frac{dx}{x}$, $v = x$.

Whence, using **U**,

$$\int \log x\,dx = x \log x - \int x \frac{dx}{x} = x \log x - x + c.$$

Because of possible error, especially while acquiring skill in using this method, the student is advised to differentiate the result obtained and thus verify its accuracy.

Ex. 2. $\int \sec^3 x\,dx$. Writing this as $\int \sec x \sec^2 x\,dx$,

let $u = \sec x$ and $dv = \sec^2 x\,dx$.

Then $du = \sec x \tan x\,dx$, $v = \tan x$.

$$\therefore \int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx.$$

Since $\tan^2 x = \sec^2 x - 1$, we may write

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$$

$$\therefore 2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$= \sec x \tan x + \log(\sec x + \tan x).$$

$$\therefore \int \sec^3 x\,dx = \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \log(\sec x + \tan x) + c.$$

Ex. 3. $\int e^{ax}\cos bx\,dx.$ Let $u = e^{ax}$ and $dv = \cos bx\,dx.$

Then $du = ae^{ax}\,dx, \qquad v = \dfrac{\sin bx}{b}.$

$$\therefore \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx. \tag{1}$$

We have thus reduced the problem of evaluating $\int e^{ax}\cos bx\,dx$ to that of evaluating $\int e^{ax}\sin bx\,dx$, which, for the moment, seems no more promising. But treating the latter in a similar way, let $u = e^{ax}$ and $dv = \sin bx\,dx.$

Then $du = ae^{ax}, \qquad v = -\dfrac{\cos b}{b}.$

$$\therefore \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx.$$

Using this value in (1),

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx.$$

$$\therefore \frac{a^2+b^2}{a^2}\int e^{ax}\cos bx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx).$$

$$\therefore \int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c.$$

Other forms of the value of the integral, involving the constant of integration differently, may be obtained as follows:

Since $\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)^2 + \left(\dfrac{a}{\sqrt{a^2+b^2}}\right)^2 = 1$, an arc α exists such that $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ and $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\alpha$. Hence, noting that $\sin\alpha\sin bx + \cos\alpha\cos bx = \cos(bx-\alpha)$, we have also

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\alpha) + c.$$

Or writing $\frac{b}{\sqrt{a^2+b^2}} = \cos \alpha$ and $\frac{a}{\sqrt{a^2+b^2}} = \sin \alpha$, another possible form is

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin(bx + \alpha).$$

It is suggested as an exercise that the student evaluate this integral by letting $u = \cos bx$ and $dv = e^{ax}dx$.

Ex. 4. $\int x^3 \sin ax\, dx$. Let $u = x^3$ and $dv = \sin ax\, dx$.

Then $du = 3\, x^2\, dx, \quad dv = -\frac{1}{a} \cos ax.$

$$\therefore \int x^3 \sin ax\, dx = -\frac{x^3}{a} \cos ax + \frac{3}{a} \int x^2 \cos ax\, dx. \tag{2}$$

Now let $u = x^2$ and $dv = \cos ax\, dx$.

Then $du = 2\, x\, dx, \quad v = \frac{2}{a} \sin ax.$

$$\therefore \int x^2 \cos ax\, dx = \frac{x^2}{a} \sin ax - \frac{2}{a} \int x \sin ax\, dz. \tag{3}$$

Finally let $u = x$ and $dv = \sin ax\, dx$.

Then $du = dx, \quad v = -\frac{1}{a} \cos ax.$

$$\therefore \int x \sin ax\, dx = -\frac{x}{a} \cos ax + \frac{1}{a} \int \cos ax\, dx$$

$$= -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax.$$

Putting this in (3), we have

$$\int x^2 \cos ax\, dx = \frac{a^2x^2 - 2}{a^3} \sin ax + \frac{2\, x}{a^2} \cos ax.$$

Using this in (2), we have finally

$$\int x^3 \sin ax\, dx = \frac{6\, x - a^2x^3}{a^3} \cos ax + \frac{3\, a^2x^2 - 6}{a^4} \sin ax + c.$$

If, instead of the above substitutions for u and dv, we had let $u = \sin ax$ and $dv = x^3\, dx$, the resulting integrand would have involved $x^4 \cos ax$, which is obviously not a simplification of the integrand we started with.

Ex. 5. $\int \sqrt{x^2 + a^2}\,dx$. Let $u = \sqrt{x^2 + a^2}$ and $dv = dx$.

Then $$du = \frac{x\,dx}{\sqrt{x^2 + a^2}}, \qquad v = x.$$

$$\therefore \int \sqrt{x^2 + a^2}\,dx = x\sqrt{x^2 + a^2} - \int \frac{x^2\,dx}{\sqrt{x^2 + a^2}}. \tag{4}$$

Using the serviceable device of writing

$$\frac{x^2}{\sqrt{x^2 + a^2}} = \frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2} - \frac{a^2}{\sqrt{x^2 + a^2}},$$

(4) takes the form

$$\int \sqrt{x^2 + a^2}\,dx = x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\,dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}.$$

$$\therefore \int \sqrt{x^2 + a^2}\,dx = \tfrac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= \tfrac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2}) + c.$$

Ex. 6. $\int x^3\sqrt{a^2 - x^2}\,dx$. Let $u = x^2$ and $dv = x\sqrt{a^2 - x^2}\,dx$.

Then $$du = 2\,x\,dx, \qquad v = -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}.$$

$$\therefore \int x^3\sqrt{a^2 - x^2}\,dx = -\frac{x^2}{3}(a^2 - x^2)^{\frac{3}{2}} + \frac{2}{3}\int (a^2 - x^2)^{\frac{3}{2}}\,x\,dx$$

$$= -\frac{x^3}{3}(a^2 - x^2)^{\frac{3}{2}} - \tfrac{2}{15}(a^2 - x^2)^{\frac{5}{2}} + c.$$

Remark. — While no general rule can be formulated as to the choice of u and dv, the fact that the application of formula U requires the evaluation of the integrals $\int dv$ and $\int v\,du$ suggests first, that dv be so selected that $\int dv$ be one of the standard forms or reducible to one by an obvious transformation; and second, that $\int v\,du$ must either be one of the standard forms or, at least, a step towards one. The illustrative exercises given above and a little practice will enable the student to use formula U effectively.

EXERCISES

Evaluate the following:

1. $\int \sin^{-1} x\, dx.$
2. $\int \tan^{-1} x\, dx.$
3. $\int \sec^{-1} x\, dx.$
4. $\int x^n \log x\, dx.$
5. $\int x \sin^{-1} x\, dx.$
6. $\int x^2 e^{ax}\, dx.$
7. $\int xe^{-x}\, dx.$
8. $\int xe^{x}\, dx.$
9. $\int x \tan^{-1} x\, dx.$
10. $\int x^2 \cos x\, dx.$
11. $\int x^2 \sin^{-1} x\, dx.$
12. $\int \sqrt{a^2 - x^2}\, dx.$
13. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}.$
14. $\int \frac{x^3\, dx}{\sqrt{x^2 + a^2}}.$
15. $\int x^2 \log^2 x\, dx.$
16. $\int e^{ax} \sin bx\, dx.$
17. $\int \sec^5 x\, dx.$
18. $\int \tan^2 x \sec x\, dx.$

53. Rational fractions. A rational fraction is the quotient of two polynomials $\frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n}$. The integration of a rational fraction is frequently effected by decomposing it into a sum of simpler partial fractions, whose denominators are factors of the denominators of the original fraction. The subject of decomposition of rational fractions is discussed fully in Algebra and will not be taken up in detail here. But the essential facts with which we shall be concerned will be enumerated and their applications given.

If the degree of the numerator is equal to or greater than that of the denominator, the fraction can be reduced to the sum of a polynomial and a fraction, by actual division, or otherwise. Thus

$$\frac{x^3 - 6x + 11}{x^2 + 2x - 3} = x - 2 + \frac{x + 5}{x^2 + 2x - 3}$$

and

$$\frac{x^4 + x^3 + 2x^2 + x}{x^4 + 2x^2 + 1} = 1 + \frac{x^3 - 1}{x^4 + 2x^2 + 1}.$$

Since the integral of the polynomial part is known, it still remains to consider the integration of the fractional part, which from now on will be assumed to have its numerator of lower degree than its denominator. Let it be $F(x) = \frac{f(x)}{\phi(x)}$.

Four cases are to be distinguished depending upon the nature of the roots of $\phi(x) = 0$. They are:

1. The roots are all real and distinct.
2. The roots are all real but some are repeated.
3. Some of the roots are imaginary, but none of these is repeated.
4. Some of the imaginary roots are repeated.

Each of these cases will be considered in turn.

Case 1. *The roots of $\phi(x) = 0$ are real and distinct.*

If the roots are $r_1, r_2, \cdots, r_n$, the denominator may be written $b_0(x - r_1)(x - r_2) \cdots (x - r_n)$ and the fraction thus takes the form $F(x) = \frac{f(x)}{b_0(x - r_1)(x - r_2) \cdots (x - r_n)}$, where $f(x)$ is surely not of higher degree than $n - 1$.

The principle of decomposition of a rational fraction into a sum of partial fractions consists, in this case, in the fact that $F(x)$ can be replaced by the sum of simpler fractions

$$F(x) = \frac{A}{x - r_1} + \frac{B}{x - r_2} + \cdots + \frac{L}{x - r_n}. \tag{1}$$

The coefficients $A, B, \cdots, L$ can be obtained by clearing of fractions and equating coefficients in the resulting identity, or by an obvious modification of this.

Integrating both members of (1), we have

$$\int F(x)\,dx$$
$$= A \log(x - r_1) + B \log(x - r_2) + \cdots + L \log(x - r_n) + c.$$

The following exercises will illustrate.

Ex. 1. $\int \frac{dx}{3x^2 - x - 2}$.

Since $3x^2 - x - 2 = (x-1)(3x+2)$, let

$$\frac{1}{3x^2 - x - 2} = \frac{A}{x-1} + \frac{B}{3x+2}.$$

Clearing of fractions,

$$1 = (3A + B)x + 2A - B.$$

Equating coefficients of like powers of x of both sides,

$$3A + B = 0, \qquad 2A - B = 1.$$

$$\therefore A = \tfrac{1}{5}, \qquad B = -\tfrac{3}{5}.$$

$$\therefore \int \frac{dx}{3x^2 - x - 2} = \frac{1}{5}\int \frac{dx}{x-1} - \frac{3}{5}\int \frac{dx}{3x+2}$$

$$= \tfrac{1}{5}\log(x-1) - \tfrac{1}{5}\log(3x+2) + c$$

$$= \log\sqrt[5]{\frac{x-1}{3x+2}} + c.$$

Ex. 2. $\int \frac{x^3 - 6x + 11}{x^2 + 2x - 3}\,dx$.

Since the numerator is of higher degree than the denominator, we have, on dividing,

$$\frac{x^3 - 6x + 11}{x^2 + 2x - 3} = x - 2 + \frac{x+5}{x^2 + 2x - 3}.$$

Noting that $x^2 + 2x - 3 = (x+1)(x-3)$, let

$$\frac{x+5}{x^2 + 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Clearing of fractions,

$$x + 5 = (A + B)x - 3A + B.$$

Equating coefficients,

$$A + B = 1, \ -3A + B = 5. \quad \therefore A = -1, \ B = 2.$$

$$\therefore \int \frac{x^3 - 6x + 11}{x^2 + 2x - 3}\,dx = \int x\,dx - 2\int dx - \int \frac{dx}{x+1} + 2\int \frac{dx}{x-3}$$

$$= \frac{x^2}{2} - 2x - \log(x+1) + 2\log(x-3) + c$$

$$= \frac{x^2}{2} - 2x + \log\frac{(x-3)^2}{x+1} + c.$$

Ex. 3. $\int \frac{2x^2 - 1}{x^3 - x}\,dx.$

Since $x^3 - x = x(x-1)(x+1)$, let

$$\frac{2x^2 - 1}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Clearing of fractions,

$$2x^2 - 1 = (A + B + C)x^2 + (B - C)x - A.$$

Equating coefficients,

$$A + B + C = 2,\ B - C = 0,\ A = +1.$$

$$\therefore A = 1,\ B = \tfrac{1}{2},\ C = \tfrac{1}{2}.$$

$$\therefore \int \frac{2x^2 - 1}{x^3 - x}\,dx = \int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= \log x + \tfrac{1}{2}\log(x-1) + \tfrac{1}{2}\log(x+1) + c$$

$$= \log x\sqrt{x^2 - 1} + c.$$

Note 1. — While the general method works nicely here, there is a possible short cut in this case that is worth mentioning. Noting that $\frac{2x^2 - 1}{x^3 - x}$ may be written $\frac{x^2 + x^2 - 1}{x(x^2 - 1)} = \frac{x}{x^2 - 1} + \frac{1}{x}$, the value of the integral follows at once.

Note 2. — A second practical suggestion may be made, this time concerning the determination of the undetermined constants A, B, C.

If on clearing of fractions we indicate the multiplication on the right-hand side, we have

$$2x^2 - 1 = (x-1)(x+1)A + x(x+1)B + x(x-1)C.$$

Assigning to x the values 0, 1, -1 in turn, we have

$$\begin{aligned} x=0: &\quad -1=-A. \quad \therefore A=1.\\ x=1: &\quad 1=2B. \quad \therefore B=\tfrac{1}{2}.\\ x=-1: &\quad 1=2C. \quad \therefore C=\tfrac{1}{2}.\end{aligned}$$

This method of determining the constants will frequently be found simpler than the other.

Ex. 4. $\displaystyle\int \frac{2x^2-x-1}{2x^3-2x^2-x}\,dx.$

Since the roots of $2x^2-2x-1$ are $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}$,

$$2x^3-2x^2-x=2x\left(x-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{3}}{2}\right).$$

In order to avoid numerical fractions, multiply numerator and denominator of the integrand by 2. Then

$$\begin{aligned}\frac{4x^2-2x-2}{4x^3-4x^2-2x} &= \frac{4x^2-2x-2}{x(2x-1-\sqrt{3})(2x-1+\sqrt{3})}\\ &= \frac{A}{x}+\frac{B}{2x-1-\sqrt{3}}+\frac{C}{2x-1+\sqrt{3}}.\end{aligned}$$

Clearing of fractions,

$$\begin{aligned}&4x^2-2x-2\\ &=(4A+2B+2C)x^2-[4A+(1-\sqrt{3})B+(1+\sqrt{3})C]x-2A.\end{aligned}$$

Equating coefficients,

$$\begin{aligned}&4A+2B+2C=4, \quad 4A+(1-\sqrt{3})B+(1+\sqrt{3})C=2,\\ &\qquad 2A=2.\end{aligned}$$

$$\therefore A=1, \quad B=\frac{1}{\sqrt{3}}, \quad C=-\frac{1}{\sqrt{3}}.$$

$$\begin{aligned}\therefore \int\frac{2x^2-x-1}{2x^3-2x^2-x}\,dx &= \int\frac{dx}{x}+\frac{1}{\sqrt{3}}\int\frac{dx}{2x-1-\sqrt{3}}\\ &\qquad -\frac{1}{\sqrt{3}}\int\frac{dx}{2x-1+\sqrt{3}}\\ &= \log x+\frac{1}{2\sqrt{3}}\log\frac{2x-1-\sqrt{3}}{2x-1+\sqrt{3}}+c.\end{aligned}$$

The actual work involved can be simplified by noting that $2x^2 - x - 1 = 2x^2 - 2x - 1 + x$; whence the integral can be replaced by $\int \frac{dx}{x} + \int \frac{dx}{2x^2 - 2x - 1}$. To evaluate the second integral, write it

$$\frac{1}{2}\int \frac{dx}{x^2 - x - \frac{1}{2}} \quad \text{or} \quad \frac{1}{2}\int \frac{dx}{(x - \frac{1}{2})^2 - \frac{3}{4}}.$$

As an exercise the student should evaluate this.

EXERCISES

Evaluate the following:

1. $\int \frac{dx}{x - x^3}$.
2. $\int \frac{x - 5}{x^2 - x - 2}\,dx$.
3. $\int \frac{x - 2}{x^3 + x^2 - x}\,dx$.
4. $\int \frac{x^4 - x^3 - 10x^2 + 9x + 4}{x^2 + x - 6}\,dx$.
5. $\int \frac{x^3 - 7x + 1}{x^2 + 2x - 3}\,dx$.
6. $\int \frac{x + 1}{x^3 - 2x^2 - x}\,dx$.
7. $\int \frac{x^2 - x - 1}{x^3 - x}\,dx$.
8. $\int \frac{x^2 - 4x + 7}{(x^2 - 1)(2 - x)}\,dx$.

Case 2. *All roots are real and some are repeated.* A root r of an equation $\phi(x) = 0$ is said to be a *k-fold root* of it if, when $\phi(x)$ is expressed as a product of factors, the factor $(x - r)$ appears to the kth power. Thus in the case of

$$\phi(x) = x^4 - x^3 - 3x^2 + 5x - 2 = (x - 1)^3(x - 2),$$

1 is a triple root, while 2 is said to be a *simple* root; and for

$$\phi(x) = 16x^4 - 8x + 1 = (2x - 1)^2(2x + 1)^2,$$

$\frac{1}{2}$ and $-\frac{1}{2}$ are double roots. Moreover, in the case of

$$\phi(x) = x^6 + x^5 - 5x^4 + 3x^3 = x^3(x - 1)^2(x + 3),$$

0 is a triple root, 1 is a double root, and -3 is a simple root.

The decomposition, in case repeated real roots appear, is effected as before, except that, corresponding to a k-fold root r, the sum of k partial fractions

$$\frac{A}{x-r}+\frac{B}{(x-r)^2}+\cdots+\frac{G}{(x-r)^k}$$

must be used.

The following exercise will illustrate.

Ex. $\displaystyle\int\frac{3x^2-8x-1}{(x-1)^3(x+2)}dx.$

Put $\displaystyle\frac{3x^2-8x-1}{(x-1)^3(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+2}.$

Clearing of fractions,

$$\begin{aligned}3x^2-8x-1=&(A+D)x^3+(B-3D)x^2\\&+(-3A+B+C+3D)x+2A-2B+2C-D.\end{aligned}$$

Equating coefficients,

$$A+D=0,\quad B-3D=3,\quad -3A+B+C+3D=-8,$$
$$2A-2B+2C-D=-1.$$

$$\therefore\ A=1,\quad B=0,\quad C=-2,\quad D=-1,\text{ and}$$

$$\begin{aligned}\int\frac{3x^2-8x-1}{(x-1)^3(x+2)}dx&=\int\frac{dx}{x-1}-2\int\frac{dx}{(x-1)^3}-\int\frac{dx}{x+2}\\&=\log(x-1)+\frac{1}{(x-1)^2}-\log(x+2)+c\\&=\log\frac{x-1}{x+2}+\frac{1}{(x-1)^2}+c.\end{aligned}$$

NOTE. — The alternative method of determining the constants A, B, C, D mentioned in Note 2 of Case 1 may also be applied here, with a slight modification.

Clearing of fractions :

$$3x^2-8x-1=(x+2)[(x-1)^2A+(x-1)B+C]+(x-1)^3D.$$

For $x=1$: $\quad -6=3C. \quad \therefore\ C=-2.$

$x=-2$: $\quad 27=-27D. \quad \therefore\ D=-1.$

These are the first values of x to select because of the obvious reduction on the right-hand side. Selecting any other two convenient values of x, (preferably such that will cause the coefficients in the right-hand member to take small values), the two resulting equations enable one to determine A and B. Thus:

For $x=0$: $\quad -1 = 2(A-B+C)-D, \quad$ or $\quad A-B=1.$

$x=2$: $\quad -5 = 4(A+B+C)+D, \quad$ or $\quad A+B=1.$

$\therefore A=1, \; B=0.$

EXERCISES

Evaluate the following:

1. $\displaystyle\int \frac{x^2+x-1}{(x-1)^3}\,dx.$

2. $\displaystyle\int \frac{dx}{x^4-4x^2+4}.$

3. $\displaystyle\int \frac{x^3+2x-4}{x^3(x+1)}\,dx.$

4. $\displaystyle\int \frac{x^4-3x^3+x^2+x-4}{x^3-4x^2+4x}\,dx.$

5. $\displaystyle\int \frac{16x^2-80x+48}{x^3(x-1)^2(x+3)}\,dx.$

6. $\displaystyle\int \frac{8x^3+4x^2-26x+3}{16x^4-8x^2+1}\,dx.$

Case 3. *Some of the roots are imaginary, but none of these is repeated.* If the coefficients of an equation are real, it can be proved that the imaginary roots can be grouped in pairs of conjugates; that is, if $\alpha + i\beta$ is a root, then $\alpha - i\beta$ is also a root, where $i=\sqrt{-1}$. Use is made of this fact to avoid the employment of imaginary expressions in the partial fractions into which the integrand is decomposed. Thus, instead of considering each of the corresponding factors of the denominator $(x-\alpha-i\beta)$ and $(x-\alpha+i\beta)$ separately, we shall consider their product

$$(x-\alpha-i\beta)(x-\alpha+i\beta)=(x-\alpha)^2+\beta^2 \quad \text{or} \quad x^2+bx+c,$$

which is real. The decomposition is effected in this case as before, except that corresponding to such a quadratic factor of the denominator we consider the partial fraction

$$\frac{Ax+B}{(x-\alpha)^2+\beta^2} \quad \text{or} \quad \frac{Ax+B}{x^2+bx+c}.$$

The following exercise will illustrate.

Ex. 1. $\displaystyle\int \frac{5-7x}{x^4-2x^3+5x^2}\,dx.$

Since it is desirable not to break up x^2-2x+5 into its linear factors, let

$$\frac{5-7x}{x^4-2x^3+5x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-2x+5}.$$

Clearing of fractions:

$$5-7x=(A+C)x^3+(-2A+B+D)x^2+(5A-2B)x+5B.$$

Equating coefficients and solving:

$$A=-1,\ B=1,\ C=1,\ D=-3.$$

$$\therefore \int \frac{5-7x}{x^4-2x^3+5x^2}\,dx=-\int\frac{dx}{x}+\int\frac{dx}{x^2}+\int\frac{x-3}{x^2-2x+5}\,dx.$$

Writing $\displaystyle\int\frac{(x-3)dx}{x^2-2x+5}=\int\frac{(x-1)dx}{x^2-2x+5}+2\int\frac{dx}{x^2-2x+5},$

we have, on integrating,

$$\int\frac{(5-7x)\,dx}{x^4-2x^3+5x^2}$$

$$=-\log x-\frac{1}{x}+\tfrac{1}{2}\log(x^2-2x+5)+\tan^{-1}\frac{x-1}{2}+c$$

$$=\log\frac{\sqrt{x^2-2x+5}}{x}-\frac{1}{x}+\tan^{-1}\frac{x-1}{2}+c.$$

NOTE. — In this discussion use is made of the fact that a polynomial with real coefficients can be represented as a product of real linear and quadratic factors arising when there are imaginary roots of the corresponding equation, obtained by equating the polynomial to zero. Similarly a polynomial with rational coefficients can often be expressed as the product of rational factors. Some of these factors will be of higher degree than the first when the roots of the corresponding equation are not all rational. This is of special interest to us here when none of the rational factors is of higher degree than the second. Thus consider the following exercise.

Ex. 2. $\int \frac{(2x-1)dx}{x^3-4x^2+2x}$. Even if the roots of $x^2-4x+2=0$ are real, since they are irrational, let

$$\frac{2x-1}{x^3-4x^2+2x}=\frac{Ax+B}{x^2-4x+2}+\frac{C}{x}.$$

$$\therefore 2x-1=(A+C)x^2+(B-4C)x+2C.$$

Equating coefficients and solving:

$$A=\tfrac{1}{2},\ B=0,\ C=-\tfrac{1}{2}.$$

$$\therefore \int\frac{(2x-1)dx}{x^3-4x^2+2x}=\frac{1}{2}\int\frac{x\,dx}{x^2-4x+2}-\frac{1}{2}\int\frac{dx}{x}$$

$$=\frac{1}{2}\int\frac{(x-2)dx}{x^2-4x+2}+\int\frac{dx}{(x-2)^2-2}-\frac{1}{2}\int\frac{dx}{x}$$

$$=\tfrac{1}{4}\log(x^2-4x+2)+\frac{1}{2\sqrt{2}}\log\frac{x-2-\sqrt{2}}{x-2+\sqrt{2}}-\tfrac{1}{2}\log x+c.$$

While in this case the irrational number $\sqrt{2}$ necessarily comes into evidence, it makes its appearance at the end only, and not early. To appreciate the point made here, the student should use the method of Case 1 in this exercise.

EXERCISES

Evaluate the following:

1. $\int\frac{2x^2+2x+1}{(x^2+1)(2x^2+1)}dx.$

2. $\int\frac{(x-4)dx}{(x^2+2)(x-1)^2}.$

3. $\int\frac{(2x^2-7)dx}{(x^2-3x+4)(x^2-3x+1)}.$

4. $\int\frac{(3x-1)dx}{(x^2+1)(x-2)}.$

5. $\int\frac{(2-2x^2)dx}{(1+x^2)(1+2x-x^2)}.$

Case 4. *Some of the imaginary roots are repeated.* If the coefficients in an equation are real, it can be proved that if the root $\alpha + i\beta$ is a k-fold root of it, $\alpha - i\beta$ also is a k-fold root. The quadratic factor[1]

$$(x-\alpha)^2+\beta^2 \quad \text{or} \quad x^2+bx+c$$

thus enters raised to the kth power. Corresponding to this factor, the decomposition requires the use of the sum of terms

$$\frac{Ax+B}{(x-\alpha)^2+\beta^2}+\frac{Cx+D}{[(x-\alpha)^2+\beta^2]^2}+\cdots+\frac{Kx+L}{[(x-\alpha)^2+\beta^2]^k}.$$

This case introduces a new type of integral that does not come directly under any of our standard forms, viz.

$$\int\frac{dz}{(z^2+\beta^2)^n}, \text{ where } z = x-\alpha.$$

To evaluate this, consider $\int\frac{dz}{(z^2+\beta^2)^{n-1}}$ and integrate by parts, letting $u=\frac{1}{(z^2+\beta^2)^{n-1}}$ and $dv = dz$. Then

$$du = -\frac{2(n-1)z\,dz}{(z^2+\beta^2)^n}, \quad v = z, \text{ and}$$

$$\int\frac{dz}{(z^2+\beta^2)^{n-1}}=\frac{z}{(z^2+\beta^2)^{n-1}}+2(n-1)\int\frac{z^2\,dz}{(z^2+\beta^2)^n}.$$

Writing $\frac{z^2}{(z^2+\beta^2)^n}=\frac{z^2+\beta^2-\beta^2}{(z^2+\beta^2)^n}=\frac{1}{(z^2+\beta^2)^{n-1}}-\frac{\beta^2}{(z^2+\beta^2)^n}$, we have

$$\int\frac{dz}{(z^2+\beta^2)^{n-1}}=\frac{z}{(z^2+\beta^2)^{n-1}}+2(n-1)\int\frac{dz}{(z^2+\beta^2)^{n-1}}$$
$$-2(n-1)\beta^2\int\frac{dz}{(z^2+\beta^2)^n}.$$

---

[1] As in the previous case, this method applies whenever it is undesirable to decompose the quadratic factor into its linear factors because these are irrational, whether real or imaginary.

Solving for the desired integral, we have the important formula[1]

$$\alpha. \quad \int \frac{dz}{(z^2+\beta^2)^n} = \frac{1}{2(n-1)\beta^2}\frac{z}{(z^2+\beta^2)^{n-1}} + \frac{2n-3}{2(n-1)\beta^2}\int\frac{dz}{(z^2+\beta^2)^{n-1}}.$$

Thus consider

$$Ex. \quad \int \frac{x^4+18x^2-28x}{(x+2)(x^2-2x+4)^2}dx.$$

Let

$$\frac{x^4+18x^2-28x}{(x+2)(x^2-2x+4)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} + \frac{Dx+E}{(x^2-2x+4)^2}.$$

Clearing of fractions,

$$x^4+18x^2-28x = (A+B)x^4 + (C-4A)x^3 + (12A+D)x^2 + (8B+2D+E-16A)x + 16A+8C+2E.$$

Equating coefficients and solving

$$A=1,\ B=0,\ C=4,\ D=6,\ E=-24.$$

$$\therefore \int \frac{x^4+18x^2-28x}{(x+2)(x^2-2x+4)^2}dx = \int\frac{dx}{x+2} + 4\int\frac{dx}{x^2-2x+4} + 6\int\frac{(x-4)\,dx}{(x^2-2x+4)^2}. \quad (1)$$

Because of the form of the reduction formula α, we shall first evaluate

$$6\int\frac{(x-4)dx}{(x^2-2x+4)^2} = 6\int\frac{(x-1)dx}{(x^2-2x+4)^2} - 18\int\frac{dx}{(x^2-2x+4)^2}. \quad (2)$$

Letting $x^2-2x+4=u$, $2(x-1)dx = du$, and

$$6\int\frac{(x-1)dx}{(x^2-2x+4)^2} = 3\int\frac{du}{u^2} = -\frac{3}{u} = -\frac{3}{x^2-2x+4}. \quad (3)$$

---

[1] A formula of this sort is known as a *reduction formula*. By means of it the problem of evaluating an integral involving the nth power of a function is replaced by one of evaluating an integral involving the function to a lower power.

Letting $x-1=z$ and noting that $x^2-2x+4=(x-1)^2+3$,

$$\int\frac{dx}{(x^2-2x+4)^2}=\int\frac{dz}{(z^2+3)^2}.$$

Applying the formula α to this, noting that $n=2$ and $\beta^2=3$,

$$\int\frac{dz}{(z^2+3)^2}=\frac{1}{6}\frac{z}{z^2+3}+\frac{1}{6}\int\frac{dz}{z^2+3}.$$

$$\therefore -18\int\frac{dx}{(x^2-2x+4)^2}=\frac{3-3x}{x^2-2x+4}-3\int\frac{dx}{x^2-2x+4}. \tag{4}$$

Noting that

$$\int\frac{dx}{x+2}=\log(x+2),$$

$$\int\frac{dx}{x^2-2x+4}=\int\frac{dx}{(x-1)^2+3}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}, \tag{5}$$

we have finally, using (2), (3), (4), (5), in (1)

$$\int\frac{x^4+18x^2-28x}{(x+2)(x^2-2x+4)^2}dx=\log(x+2)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}$$
$$-\frac{3x}{x^2-2x+4}+c.$$

EXERCISES

Evaluate the following:

1. $\int\frac{x^3\,dx}{(x^2+4)^2}.$

2. $\int\frac{2x\,dx}{(x+1)(x^2+1)^2}.$

3. $\int\frac{2x^3+3x^2-5x+2}{(x^2-2)^3}dx.$

4. $\int\frac{x^2\,dx}{(x^2+1)^3}.$

54. Functions containing radicals; algebraic transformation. In the previous section we showed how the integration of any rational function of a variable can be reduced to the integration of simpler functions which come under one or another of the standard forms. Most of the remaining methods of integration consist in the introduction of a new variable which transforms the integrand either into a rational function of the new variable or into a known form.

Several important and frequently occurring cases will be mentioned here.

A. The function contains fractional powers of $a + bx$ (where a may be zero) *and no other radicals.*

If the least common denominator of the exponents of $a + bx$ is k, the transformation

$$a + bx = u^k,$$

carrying with it $dx = \frac{ku^{k-1}}{b} du$, will lead to a rational function of u. Thus consider

Ex. 1. $\int \frac{x\,dx}{(2+x)^{\frac{2}{3}}}$. Let $2 + x = u^3$. Then $dx = 3u^2\,du$.

$$\therefore 3\int \frac{(u^3-2)u^2\,du}{u^2} = 3\int (u^3-2)du = \tfrac{3}{4}u^4 - 6u + c$$
$$= \tfrac{3}{4}(2+x)^{\frac{4}{3}} - 6(2+x)^{\frac{1}{3}} + c.$$

Ex. 2. $\int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}$. Let $x = u^6$. Then $dx = 6u^5\,du$.

$$\therefore \int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}} = 6\int \frac{u^5\,du}{u^3 - u^2} = 6\int \frac{u^3\,du}{u-1} = 6\int \frac{u^3 - 1 + 1}{u-1}du$$
$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right)du$$
$$= 2u^3 + 3u^2 + 6u + 6\log(u-1) + c$$
$$= 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} + \log(x^{1} - 1)^6 + c.$$

B. The function contains the single radical $\sqrt{ax^2 + bx + c}$.

(*a*) *If* a *is positive* let $\sqrt{ax^2 + bx + c} = u - x\sqrt{a}$.

Then $$ax^2 + bx + c = u^2 - 2ux\sqrt{a} + ax^2.$$

$$\therefore x = \frac{u^2 - c}{b + 2u\sqrt{a}} \quad \text{and} \quad dx = \frac{2(u^2\sqrt{a} + bu + c\sqrt{a})}{(b + 2u\sqrt{a})^2}du.$$

The transformed integrand is thus rational and real.

This method was used in obtaining the formula **P.**

(*b*) *If a is negative and c is positive,* let

$$\sqrt{ax^2+bx+c} = ux - \sqrt{c}.$$

Then $$ax^2+bx+c = u^2x^2 - 2\,ux\sqrt{c} + c.$$

$$\therefore x = \frac{b+2\,u\sqrt{c}}{u^2-a} \quad \text{and} \quad dx = -\frac{2(u^2\sqrt{c}+bu+a\sqrt{c})}{(u^2-a^2)^2}\,du.$$

The transformed integrand is thus rational and real.

Noting that, in this case, the roots of $ax^2+bx+c=0$ are real, say α and β, we may write

$$\sqrt{ax^2+bx+c} = \sqrt{-a}\sqrt{(x-\alpha)(\beta-x)}.$$

This suggests an alternative transformation $\sqrt{\dfrac{x-\alpha}{\beta-x}} = u$. Then

$$x = \frac{\beta u^2+\alpha}{u^2+1},\ dx = \frac{2(\beta-\alpha)u}{(u^2+1)^2},\ \text{and}$$

$$\sqrt{ax^2+bx+c} = \sqrt{-a}\,u(\beta-x) = \frac{\sqrt{-a}\,u(\beta-\alpha)}{u^2+1}.$$

The transformed integrand is thus rational and real.

(*c*) *If both a and c are negative* and the roots α and β are real, the second transformation in (*b*) is still available. But if α and β are imaginary, there is no transformation that will lead to a rational integrand that is real.

Remark. — The transformation $\frac{1}{x} = u$, leads to the following simplifications:

$$\int\frac{dx}{x\sqrt{ax^2+bx+c}} = -\int\frac{du}{\sqrt{a+bu+cu^2}},$$

$$\int\frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\int\frac{u\,du}{\sqrt{a+bu+cu^2}}.$$

It was this *reciprocal substitution* that suggested the first transformation in (*b*), when the transformation in (*a*) leads to imaginary coefficients.

Ex. 3. $$\int\frac{dx}{x\sqrt{8x^2+6x+1}}=-\int\frac{du}{\sqrt{8+6u+u^2}}$$

$$=-\int\frac{du}{\sqrt{(u+3)^2-1}},\ \left[\text{here } \frac{1}{x}=u\right]$$

$$=-\log\left(u+3+\sqrt{u^2+6x+8}\right)+c$$

$$=\log\frac{1}{u+3+\sqrt{u^2+6x+8}}\cdot\frac{u+3-\sqrt{u^2+6x+8}}{u+3-\sqrt{u^2+6x+8}}+c$$

$$=\log\frac{u+3-\sqrt{u^2+6x+8}}{1}+c$$

$$=\log\frac{3x+1-\sqrt{8x^2+6x+1}}{x}+c.$$

Ex. 4. $$\int\frac{dx}{x^2\sqrt{x^2+4}}=-\int\frac{u\,du}{\sqrt{1+4u^2}},\ \text{where } \frac{1}{x}=u.$$

Letting $1+4u^2=t,\ u\,du=\frac{dt}{8}\cdot$

$$\therefore\int\frac{dx}{x^2\sqrt{x^2+4}}=-\frac{1}{8}\int\frac{dt}{\sqrt{t}}=-\frac{\sqrt{t}}{4}+c=-\frac{\sqrt{1+u^2}}{4}+c$$

$$=-\frac{\sqrt{x^2+4}}{4x}+c.$$

C. $\int x^m(a+bx^n)^p\,dx$, *where* m, n, p *are rational numbers.* If m and n are not integers, let k be their least common denominator. Then, by A, the transformation $x=u^k$ leads to $k\int u^{mk+k-1}(ax+bu^{nk})^p\,du$, where $mk+k-1$ and nk are integers. So the only further case to be considered is where m and n are integers and p is a fraction. The integral to be evaluated then is

$$\int x^m(a+bx^n)^{\frac{r}{s}}\,dx,\ \textit{where } m,\ n,\ r,\ s \textit{ are integers.}$$

Let $a + bx^n = u^s$. Then $(a + bx^n)^{\frac{r}{s}} = u^r$, $x = \left(\frac{u^s - a}{b}\right)^{\frac{1}{n}}$,

$$dx = \frac{su^{s-1}}{nb}\left(\frac{u^s - a}{b}\right)^{\frac{1}{n}-1}.$$

$$\therefore \int x^m(a + bx^n)^{\frac{r}{s}}\,dx = \frac{s}{nb}\int u^{r+s-1}\left(\frac{u^s - a}{b}\right)^{\frac{m+1}{n}-1} du.$$

We have thus shown that

(*a*) *If* $\frac{m+1}{n}$ *is an integer or zero, the transformation* $a + bx^n = u^s$ *rationalizes the resulting integrand.*

Now let $a + bx^n = x^n u^s$. Then $x = a^{\frac{1}{n}}(u^s - b)^{-\frac{1}{n}}$, $(a + bx^n)^{\frac{r}{s}}$ $= a^{\frac{r}{s}}u^r(u^s - b)^{-\frac{r}{s}}$ and $dx = -\frac{sa^{\frac{1}{n}}}{n}u^{s-1}(u^s - b)^{-\frac{1}{n}-1}du.$

$$\therefore \int x^m(a + bx^{n\frac{r}{s}}dx = -\frac{sa^{\frac{m+1}{n}+\frac{r}{s}}}{n}\int u^{r+s-1}(u^s - b)^{-\frac{m+1}{n}-\frac{r}{s}-1}du.$$

We have now

(*b*) *If* $\frac{m+1}{n} + \frac{r}{s}$ *is an integer or zero, the transformation* $a + bx^n = x^n u^s$ *rationalizes the resulting integrand.*

Ex. 5. $\int\frac{(1 + x^3)^{\frac{2}{3}}}{x}dx.$ Here

$$m = -1,\ n = 3,\ r = 2,\ s = 3. \quad \therefore \frac{m+1}{n} = 0.$$

Hence we have case (*a*). Let $1 + x^3 = u^3$. Then $(1 + x^3)^{\frac{2}{3}} = u^2$, $x^3 = u^3 - 1$. $\therefore x^2\,dx = u^2\,du$ and $\frac{dx}{x} = \frac{u^2\,du}{u^3 - 1}$.

$$\therefore \int \frac{(1+x^3)^{\frac{2}{3}}}{x}\,dx = \int \frac{u^4\,du}{u^3-1} = \int u\,du + \int \frac{u\,du}{u^3-1}.$$

$$\frac{u}{u^3-1} = \frac{1}{3}\frac{1}{u-1} - \frac{1}{3}\frac{u-1}{u^2+u+1}$$

$$= \frac{1}{3}\frac{1}{u-1} - \frac{1}{3}\frac{u+\frac{1}{2}}{u^2+u+1} + \frac{1}{2}\frac{1}{u^2+u+1}.$$

$$\therefore \int \frac{(1+x^3)^{\frac{2}{3}}}{x}\,dx = \frac{u^2}{2} + \tfrac{1}{3}\log(u-1) - \tfrac{1}{6}\log(u^2+u+1)$$
$$+ \frac{1}{\sqrt{3}}\tan^{-1}\frac{2u+1}{\sqrt{3}} + c$$

$$= \frac{(1+x^3)^{\frac{2}{3}}}{2} + \tfrac{1}{3}\log\frac{(1+x^3)^{\frac{1}{3}}-1}{\sqrt{(1+x^3)^{\frac{2}{3}}+(1+x^3)^{\frac{1}{3}}+1}}$$
$$+ \frac{1}{\sqrt{3}}\tan^{-1}\frac{2(1+x^3)^{\frac{1}{3}}-1}{\sqrt{3}} + c.$$

Ex. 6. $\int \frac{dx}{x^3(1+x^4)^{\frac{1}{2}}}$. Here

$$m=-3,\ n=4,\ \frac{r}{s}=-\frac{1}{2}.\quad \therefore\ \frac{m+1}{n}+\frac{r}{s}=-\frac{2}{4}-\frac{1}{2}=-1.$$

Hence we have case (*b*). Let $1+x^4=x^4u^2$. Then

$$x^4=(u^2-1)^{-1},\quad x=(u^2-1)^{-\frac{1}{4}},\quad dx=-\tfrac{1}{2}u(u^2-1)^{-\frac{5}{4}}\,du,$$
$$1+x^4=u^2(u^2-1)^{-1}.$$

$$\therefore \int \frac{dx}{x^3(1+x^4)^{\frac{1}{2}}} = -\frac{1}{2}\int du = -\frac{u}{2}+c = -\frac{\sqrt{1+x^4}}{2x^2}+c.$$

EXERCISES

Evaluate the following:

1. $\int \frac{(x+3)^{\frac{3}{4}}-2}{(x+3)^{\frac{1}{2}}-1}\,dx.$

2. $\int \frac{3x-4}{x\sqrt{x+1}}\,dx.$

3. $\int \frac{dx}{(1-x)\sqrt{x}}.$

4. $\int \frac{dx}{x^{\frac{2}{3}}+3x^{\frac{3}{4}}}.$

5. $\displaystyle\int\frac{4\,dx}{x\sqrt{x^2-2\,x+3}}$.

6. $\displaystyle\int\frac{(x+2)^{\frac{1}{2}}+1}{(x+2)^{\frac{1}{3}}+1}\,dx$.

7. $\displaystyle\int\frac{2\,dx}{\sqrt{5\,x-6-x^2}}$.

8. $\displaystyle\int\frac{\sqrt{2\,x+3}}{2\,x-3}\,dx$.

9. $\displaystyle\int\frac{dx}{x\sqrt{25\,x^2-1}}$.

10. $\displaystyle\int\frac{\sqrt{4-x^2}}{x^4}\,dx$.

11. $\displaystyle\int\frac{dx}{x\sqrt{9-24\,x-5\,x^2}}$.

12. $\displaystyle\int x^5(a+x^3)^{-\frac{1}{3}}\,dx$.

13. $\displaystyle\int\frac{dx}{(10\,x+1)\sqrt{x^2+x}}$.

14. $\displaystyle\int\frac{dx}{x^2\sqrt{27\,x^2+6\,x-1}}$.

15. $\displaystyle\int\frac{dx}{x(1+x^5)^{\frac{1}{4}}}$.

16. $\displaystyle\int\frac{dx}{(1-x^3)^{\frac{1}{3}}}$.

55. Trigonometric transformation.

A. If integral powers of $\sqrt{a^2-x^2}$ are the only radicals appearing in the integrand, the transformation $x=a\sin\theta$ or $x=a\cos\theta$ will cause the resulting integrand to be rational.

B. If integral powers of $\sqrt{x^2-a^2}$ are the only radicals appearing, the transformation $x=a\sec\theta$ or $x=a\csc\theta$ is to be made.

C. If integral powers of $\sqrt{x^2+a^2}$ are the only radicals appearing, the transformation $x=a\tan\theta$ or $x=a\cot\theta$ is to be made.

Ex. 1. $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x}dx$. Let $x=a\sin\theta$; then $dx=a\cos\theta\,d\theta$, and $\sqrt{a^2-x^2}=a\cos\theta$.

$$\begin{aligned}\therefore\int\frac{\sqrt{a^2-x^2}}{x}\,dx&=a\int\frac{\cos^2\theta\,d\theta}{\sin\theta}=a\int\frac{1-\sin^2\theta}{\sin\theta}\,d\theta\\&=a\int(\csc\theta-\sin\theta)d\theta\\&=a\log(\csc\theta-\cot\theta)+a\cos\theta+c\\&=a\log\left(\frac{a-\sqrt{a^2-x^2}}{x}\right)+\sqrt{a^2-x^2}+c.\end{aligned}$$

Ex. 2. $\displaystyle\int\frac{\sqrt{(x^2-a^2)^3}}{x}\,dx.$ Let $x=a\sec\theta$; then

$dx=a\sec\theta\tan\theta\,d\theta$, and $\sqrt{(x^2-a^2)^3}=a^3\tan^3\theta$.

$$\begin{aligned}\therefore\int\frac{\sqrt{(x^2-a^2)^3}}{x}\,dx &= a^3\int\tan^4\theta\,d\theta = a^3\int\tan^2\theta(\sec^2\theta-1)d\theta\\ &= a^3\int(\tan^2\theta\sec^2\theta-\tan^2\theta)d\theta\\ &= a^3\int(\tan^2\theta\sec^2\theta-\sec^2\theta+1)d\theta\\ &= \frac{a^3\tan^3\theta}{3}-a^3\tan\theta+a^3\theta+c\\ &= \tfrac{1}{3}\sqrt{(x^2-a^2)^3}-a^2\sqrt{x^2-a^2}+a^3\sec^{-1}\frac{x}{a}+c\\ &= \tfrac{1}{3}\sqrt{(x^2-a^2)^3}-a^2\sqrt{x^2-a^2}+a^3\cos^{-1}\frac{a}{x}+c.\end{aligned}$$

Ex. 3. $\displaystyle\int x^3\sqrt{x^2+a^2}dx.$ Let $x=a\tan\theta$; then $dx=a\sec^2\theta\,d\theta$, and $\sqrt{x^2+a^2}=a\sec\theta$.

$$\begin{aligned}\therefore\int x^3\sqrt{x^2+a^2}\,dx &= a^5\int\tan^3\theta\sec^3\theta\,d\theta\\ &= a^5\int(\sec^2\theta-1)\sec^2\theta\sec\theta\tan\theta\,d\theta\\ &= a^5\left(\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}\right)+c\\ &= \tfrac{1}{5}\sqrt{(x^2+a^2)^5}-\frac{a^2}{3}\sqrt{(x^2+a^2)^3}+c.\end{aligned}$$

Ex. 4. $\displaystyle\int\sqrt{2\,ax-x^2}\,dx.$ Since

$$2\,ax-x^2=a^2-a^2+2\,ax-x^2=a^2-(x-a)^2,$$

let $x-a=a\sin\theta$; then $dx=a\cos\theta\,d\theta$, $\sqrt{2\,ax-x^2}=a\cos\theta$.

$$\therefore \int \sqrt{2\,ax - x^2}\,dx = a^2 \int \cos^2\theta\,d\theta = \frac{a^2}{2}\int (1 + \cos 2\,\theta)\,d\theta$$

$$= \frac{a^2\theta}{2} + \frac{a^2 \sin 2\,\theta}{4} + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + \frac{(x-a)\sqrt{2\,ax - x^2}}{2} + c,$$

since $a^2 \sin 2\,\theta = 2\,a^2 \sin\theta\cos\theta = 2(x-a)\sqrt{2\,ax - x^2}$.

Remark 1. — The methods of this section can be extended to cases where the only irrationalities are integral powers of

$$\sqrt{a^{\frac{2}{s}} - x^{\frac{2}{s}}}, \qquad \sqrt{a^{\frac{2}{s}} + x^{\frac{2}{s}}}, \qquad \sqrt{x^{\frac{2}{s}} - a^{\frac{2}{s}}},$$

$$\sqrt{a^{\frac{1}{s}} - x^{\frac{1}{s}}}, \qquad \sqrt{a^{\frac{1}{s}} + x^{\frac{1}{s}}}, \qquad \sqrt{x^{\frac{1}{s}} - a^{\frac{1}{s}}},$$

s being an integer. In such cases, the respective transformations are

$$x = a\sin^s\theta, \qquad x = a\tan^s\theta, \qquad x = a\sec^s\theta,$$

$$x = a\sin^{2s}\theta, \qquad x = a\tan^{2s}\theta, \qquad x = a\sec^{2s}\theta.$$

Ex. 5. $\int x\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\,dx$. Let $x = a\sin^3\theta$; then

$$dx = 3\,a\sin^2\theta\cos\theta\,d\theta \quad \text{and} \quad \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}} = a^{\frac{1}{3}}\cos\theta.$$

$$\therefore \int x\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\,dx = 3\,a^{\frac{7}{3}}\int \sin^5\theta\cos^2\theta\,d\theta$$

$$= 3\,a^{\frac{7}{3}}\int \sin^4\theta\cos^2\theta\sin\theta\,d\theta.$$

Noting that $\sin^4\theta = (\sin^2\theta)^2 = (1 - \cos^2\theta)^2$

$$= 1 - 2\cos^2\theta + \cos^4\theta,$$

and that $d\cos\theta = -\sin\theta\,d\theta$,

$$\int x\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\,dx = 3\,a^{\frac{7}{3}}\int (\cos^2\theta - 2\cos^4\theta + \cos^6\theta)\sin\theta\,d\theta$$

$$= -a^{\frac{7}{3}}\cos^3\theta + \tfrac{6}{5}\,a^{\frac{7}{3}}\cos^5\theta - \tfrac{3}{7}\,a^{\frac{7}{3}}\cos^7\theta + c$$

$$= -a^{\frac{4}{3}}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} + \tfrac{6}{5}\,a^{\frac{2}{3}}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{5}{2}} - \tfrac{3}{7}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{7}{2}} + c.$$

Ex. 6. $\int \sqrt{a^{\frac{1}{3}} + x^{\frac{1}{3}}}\, dx.$ Let $x = a \tan^6 \theta$; then

$$dx = 6\, a \tan^5 \theta \sec^2 \theta\, d\theta \quad \text{and} \quad \sqrt{a^{\frac{1}{3}} + x^{\frac{1}{3}}} = a^{\frac{1}{6}} \sec \theta.$$

$$\therefore \int \sqrt{a^{\frac{1}{3}} + x^{\frac{1}{3}}}\, dx = 6\, a^{\frac{7}{6}} \int \tan^5 \theta \sec^3 \theta\, d\theta$$

$$= 6\, a^{\frac{7}{6}} \int \tan^4 \theta \sec^2 \theta \sec \theta \tan \theta\, d\theta.$$

Noting that $\tan^4 \theta \sec^2 \theta = (\sec^2 \theta - 1)^2 \sec^2 \theta$

$$= \sec^6 \theta - 2 \sec^4 \theta + \sec^2 \theta,$$

and $$d \sec \theta = \sec \theta \tan \theta\, d\theta,$$

$$\int \sqrt{a^{\frac{1}{3}} + x^{\frac{1}{3}}}\, dx = 6\, a^{\frac{7}{6}} \int (\sec^6 \theta - 2 \sec^4 \theta + \sec^2 \theta)\, d \sec \theta$$

$$= \tfrac{6}{7}\, a^{\frac{7}{6}} \sec^7 \theta - \tfrac{12}{5}\, a^{\frac{7}{6}} \sec^5 \theta + 2\, a^{\frac{7}{6}} \sec^3 \theta + c$$

$$= \tfrac{6}{7} (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{\frac{7}{2}} - \tfrac{12}{5}\, a^{\frac{1}{3}} (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{\frac{5}{3}} + 2\, a^{\frac{2}{3}} (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{\frac{3}{2}} + c.$$

Remark 2. — Frequently it is possible to rationalize two distinct radicals by means of a single trigonometric transformation. Thus consider

Ex. 7. $\int \frac{\sqrt{x}}{\sqrt{a - x}}\, dx.$ Let $x = a \sin^2 \theta$; then $dx = 2\, a \sin \theta \cos \theta\, d\theta$,

$$\sqrt{x} = \sqrt{a} \sin \theta, \quad \text{and} \quad \sqrt{a - x} = \sqrt{a} \cos \theta.$$

$$\therefore \int \frac{\sqrt{x}}{\sqrt{a - x}}\, dx = 2\, a \int \sin^2 \theta\, d\theta = a \int (1 - \cos 2\,\theta)\, d\theta = a\theta - \frac{a}{2} \sin 2\,\theta + c$$

$$= a \sin^{-1} \sqrt{\frac{x}{a}} - \sqrt{ax - x^2} + c,$$

since $$\sin 2\,\theta = 2 \sin \theta \cos \theta = 2 \sqrt{\frac{x}{a}} \sqrt{\frac{a - x}{a}} = \frac{2}{a} \sqrt{ax - x^2}.$$

Ex. 8. $\int \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x + a}}\, dx.$ Let $x = a \tan^2 \theta$; then

$$dx = 2\, a \tan \theta \sec^2 \theta\, d\theta, \quad \sqrt{x} = \sqrt{a} \tan \theta \quad \text{and} \quad \sqrt{x + a} = \sqrt{a} \sec \theta.$$

$$\therefore \int \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x + a}}\, dx = 2\, a \int \frac{\tan \theta - 1}{\sec \theta} \tan \theta \sec^2 \theta\, d\theta$$

$$= 2\, a \int \tan^2 \theta \sec \theta\, d\theta - 2\, a \int \tan \theta \sec \theta\, d\theta.$$

$$\int \tan^2 \theta \sec \theta\, d\theta = \int \sec^3 \theta\, d\theta - \int \sec \theta\, d\theta.$$

In § 52 Ex. 2 we found

$$\int \sec^3 \theta\, d\theta = \tfrac{1}{2} \sec \theta \tan \theta + \tfrac{1}{2} \log (\sec \theta + \tan \theta).$$

$$\therefore \int \tan^2 \theta \sec \theta\, d\theta = \tfrac{1}{2} \sec \theta \tan \theta - \tfrac{1}{2} \log (\sec \theta + \tan \theta).$$

$$\begin{aligned} \therefore \int \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x + a}}\, dx & \\ &= a \sec \theta \tan \theta - a \log (\sec \theta + \tan \theta) - 2\, a \sec \theta + c \\ &= \sqrt{x^2 + ax} - a \log \frac{\sqrt{x + a} + \sqrt{x}}{\sqrt{a}} - 2\sqrt{ax + a^2} + c \\ &= \sqrt{x^2 + ax} - a \log (\sqrt{x + a} + \sqrt{x}) - 2\sqrt{ax + a^2} + c', \end{aligned}$$

where $c' = a \log \sqrt{a} + c$.

Remark 3. — In case the integrand is a rational function involving $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$, the transformation $x = a \sin \theta$, $x = a \tan \theta$, or $x = a \sec \theta$ respectively, will frequently be found serviceable.

EXERCISES

Evaluate the following

1. $\displaystyle\int \frac{\sqrt{4 - x^2}}{x^4}\, dx.$
2. $\displaystyle\int \frac{dx}{x^4 \sqrt{x^2 + 4}}.$
3. $\displaystyle\int \frac{dx}{x^3 \sqrt{x^2 - 1}}.$
4. $\displaystyle\int x^3 \sqrt{2 - x^2}\, dx.$
5. $\displaystyle\int \frac{dx}{x\sqrt{a^2 - x^2}}.$
6. $\displaystyle\int \sqrt{x^2 + 2\,x}\, dx.$
7. $\displaystyle\int \frac{dx}{(x^2 - 1)^{\frac{3}{2}}}.$
8. $\displaystyle\int \frac{dx}{x^2 (1 + x^2)^{\frac{3}{2}}}.$
9. $\displaystyle\int \frac{x^3\, dx}{(1 - x^2)^{\frac{3}{2}}}.$
10. $\displaystyle\int \frac{a\, dx}{x\sqrt{a^2 + x^2}}.$
11. $\displaystyle\int \frac{x\, dx}{\sqrt{a^{\frac{1}{2}} - x^{\frac{1}{2}}}}.$
12. $\displaystyle\int \frac{\sqrt{x^{\frac{2}{3}} + a^{\frac{2}{3}}}}{x^{\frac{1}{3}}}\, dx.$
13. $\displaystyle\int \frac{x^3\, dx}{(4 - x^2)^5}.$
14. $\displaystyle\int \frac{x^3\, dx}{(x^2 + 5)^3}.$
15. $\displaystyle\int \frac{x^5\, dx}{(x^2 - 3)^4}.$

56. Change of variable in definite integral. In evaluating an indefinite integral by means of a transformation of variable, we first found the integral in terms of the new variable, and then expressed this result in terms of the original variable. The process is simpler in the case of definite integrals.

If the change of variable $x = \psi(u)$ transforms $f(x)dx$ into $\phi(u)du$, and if $u = u_1$ when $x = x_1$, and $u = u_2$ when $x = x_2$, it can be proved that

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} \phi(u)\,du.$$

Hence, if in the course of evaluating a definite integral we have occasion to introduce a new variable, it is not necessary to return to the original variable.

The following exercises will illustrate:

Ex. 1. $\int_{\frac{a}{2}}^{a} \frac{\sqrt{a^2 - x^2}}{x}\,dx.$ In Ex. 1, § 55, we saw that the introduction of the new variable $x = a \sin\theta$ or $\theta = \sin^{-1}\frac{x}{a}$ led to the integral $a\int(\csc\theta - \sin\theta)d\theta = a\log(\csc\theta - \cot\theta) + a\cos\theta$. Noting that when $x = \frac{a}{2}$, $\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, and when $x = a$, $\theta = \sin^{-1} 1 = \frac{\pi}{2}$, we can write

$$\begin{aligned}\int_{\frac{a}{2}}^{a} \frac{\sqrt{a^2 - x^2}}{x}\,dx &= a\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc\theta - \sin\theta)d\theta \\ &= a\log(\csc\theta - \cot\theta) + a\cos\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= -a\log(2 - \sqrt{3}) - \frac{a}{2}\sqrt{3} = a\log(2 + \sqrt{3}) - \frac{a}{2}\sqrt{3}.\end{aligned}$$

Ex. 2. $\int_{a}^{2a} \frac{\sqrt{(x^2 - a^2)^3}}{x}\,dx.$ In Ex. 2, § 55, we saw that the introduction of the new variable $x = a\sec\theta$ or $\theta = \sec^{-1}\frac{x}{a}$ led to $a^3\int\tan^4\theta\,d\theta = \frac{a^3}{3}\tan^3\theta - a^3\tan\theta + a^3\theta$. Noting that when

$x = a$, $\theta = \sec^{-1} 1 = 0$, and when $x = 2a$, $\theta = \sec^{-1} 2 = \frac{\pi}{3}$, we can write

$$\int_a^{2a} \frac{\sqrt{(x^2 - a^2)^3}}{x} dx = 4\int_0^{\frac{\pi}{3}} \tan^4 \theta \, d\theta$$

$$= \frac{a^3}{3} \tan^3 \theta - a^3 \tan \theta + a^3 \theta \Big]_0^{\frac{\pi}{3}} = \frac{a^3 \pi}{3}.$$

EXERCISES

Evaluate the following

1. $\int_0^2 \sqrt{(4 - x^2)^3}\, dx.$
2. $\int_0^a \frac{x^2\, dx}{\sqrt{a^2 - x^2}}.$
3. $\int_0^3 \frac{dx}{(9 + x^2)^{\frac{3}{2}}}.$
4. $\int_1^3 \sqrt{4 - (x - 1)^2}\, dx.$
5. $\int_4^5 \frac{dx}{x^2 \sqrt{x^2 - 16}}.$
6. $\int_0^a \sqrt{a^2 - x^2}\, dx.$

57. Rational function of the trigonometric functions.[1] In § 55 we showed that the integration of a function involving certain commonly occurring radicals can be reduced to the integration of a rational function of the trigonometric functions by the introduction of an appropriate new variable.

While in many cases actually arising the integration of such a function of the trigonometric functions can be effected by more or less obvious simple devices (some of which will be given in §§ 58 and 59), it may be worth calling attention to the fact that *in the case of any rational function of the trigonometric functions of a variable x, the introduction of the new variable* $t = \tan \frac{x}{2}$ *results in a rational function of t.* This follows from the fact that each of the trigonometric functions of x as well as $\frac{dx}{dt}$ is a rational function of t. Thus noting that

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[1] Throughout this chapter the expression "the trigonometric functions" means the elementary trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant.

$$\sin\frac{x}{2}=\frac{\tan\frac{x}{2}}{\sqrt{1+\tan^2\frac{x}{2}}},\ \cos\frac{x}{2}=\frac{1}{\sqrt{1+\tan^2\frac{x}{2}}},\ \text{we have}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}=\frac{1+\tan^3\frac{x}{2}}{2\tan\frac{x}{2}}=\frac{2t}{1+t^2},$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

$$\therefore\ \tan x=\frac{2t}{1-t^2},\quad \cot x=\frac{1-t^2}{2t},\quad \sec\theta=\frac{1+t^2}{1-t^2},\quad \csc\theta=\frac{1+t^2}{2t}.$$

Moreover, on differentiating $\tan\frac{x}{2}=t$ or $x=2\tan^{-1}t$,

$$dx=\frac{2\,dt}{1+t^2}.$$

The following exercises will illustrate:

Ex. 1. $\int\csc x\,dx$. Letting $\tan\frac{x}{2}=t$,

$$\csc x=\frac{1+t^2}{2t},\quad dx=\frac{2\,dt}{1+t^2}.$$

$$\therefore\int\csc x\,dx=\int\frac{dt}{t}=\log t+c=\log\tan\frac{x}{2}+c. \qquad (1)$$

NOTE. — Our standard form T gave the integral of $\csc x$ as $\log(\csc x-\cot x)$. That these two results are consistent follows from the fact that

$$\tan\frac{x}{2}=\sqrt{\frac{1-\cos x}{1+\cos x}}=\frac{1-\cos x}{\sin x}=\csc x-\cot x.$$

But in many instances, the value of the integral in terms of the half angle leads to a simpler result; hence the alternate form is worth remembering.

If in (1) we let $x = u + \frac{\pi}{2}$, since $\csc\left(u + \frac{\pi}{2}\right) = \sec u$, we have

$$\int \sec u\, du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + c, \tag{2}$$

which is also worth noting.

Ex. 2. $\int \frac{dx}{5 + 4\cos x}$. Letting $\tan \frac{x}{2} = t$,

$$\int \frac{dx}{5 + 4\cos x} = 2\int \frac{dt}{t^2 + 9} = \tfrac{2}{3}\tan^{-1}\frac{t}{3} + c = \tfrac{2}{3}\tan^{-1}\left(\tfrac{1}{3}\tan\frac{x}{2}\right) + c.$$

Ex. 3. $\int \frac{a\sin x + b\cos x}{m\sin x + n\cos x}\,dx$. Letting $\tan\frac{x}{2} = t$, we are led to

$$I \equiv 2\int \frac{(2\,at + b - bt^2)\,dt}{(2\,mt + n - nt^2)(1 + t^2)}.$$

Putting $\frac{2\,b + 4\,at - 2\,bt^2}{(n + 2\,mt - nt^2)(1 + t^2)} = \frac{A + Bt}{n + 2\,mt - mt^2} + \frac{C + Dt}{1 + t^2}$

we find, on clearing of fractions, equating coefficients, and solving,

$$A = \frac{2\,m(bm - an)}{m^2 + n^2}, \qquad B = \frac{2\,n(an - bm)}{m^2 + n^2},$$

$$C = \frac{2(am + bn)}{m^2 + n^2}, \qquad D = \frac{2(an - bm)}{m^2 + n^2}.$$

$$\therefore I = \frac{bm - an}{m^2 + n^2}\int \frac{2\,m - 2\,nt}{n + 2\,mt - nt^2}\,dt$$

$$+ \frac{2}{m^2 + n^2}\int \frac{am + bn + (an - bm)t}{1 + t^2}\,dt$$

$$= \frac{bm - an}{m^2 + n^2}\log(n + 2\,mt - nt^2) + \frac{2(am + bn)}{m^2 + n^2}\tan^{-1} t$$

$$+ \frac{an - bm}{m^2 + n^2}\log(1 + t^2) + c$$

$$= \frac{bm - an}{m^2 + n^2}\log\frac{2\,mt + n(1 - t^2)}{1 + t^2} + \frac{2(am + bn)}{m^2 + n^2}\tan^{-1} t + c.$$

Remembering that $\dfrac{2t}{1+t^2} = \sin x$ and $\dfrac{1-t^2}{1+t^2} = \cos x$, we have, on passing back to the original variable,

$$\int \frac{a \sin x + b \cos x}{m \sin x + n \cos x}\, dx = \frac{bm - an}{m^2 + n^2} \log (m \sin x + n \cos x) + \frac{am + bn}{m^2 + n^2} x + c.$$

EXERCISES

Evaluate the following:

1. $\displaystyle\int \frac{dx}{4 + 5 \cos x}$.
2. $\displaystyle\int \frac{dx}{5 + 4 \sin x}$.
3. $\displaystyle\int \frac{dx}{4 + 5 \sin x}$.
4. $\displaystyle\int \frac{dx}{3 \cos x - 4 \sin x}$.
5. $\displaystyle\int \frac{2\, dx}{1 + \tan x}$.
6. $\displaystyle\int \frac{dx}{\sin x + \cos x + 1}$.
7. $\displaystyle\int \frac{dx}{\sin x - \cos x + 1}$.
8. $\displaystyle\int \frac{dx}{\sin x + \tan x}$.
9. $\displaystyle\int \frac{dx}{2 \csc x - \sin x}$.
10. $\displaystyle\int \frac{(1 + \sin x)\, dx}{\sin x(1 + \cos x)}$.

58. Other methods for functions of the trigonometric functions. While the method of the previous section works always, theoretically, it may prove difficult in practice if the resulting rational fraction is complicated. Other methods for the integration of some functions of the trigonometric functions will frequently be found more practical. These depend largely upon the fact that the derivative of each of the trigonometric functions is given in terms of another one with which it is connected by a simple quadratic relation. Illustrations of such types of functions were given in the exercises of § 55. Without attempting to exhaust the list, the following cases may be enumerated.

A. $\int f(\sin x, \cos x)\cos x\,dx.$

Since $d\sin x = \cos x\,dx$ and $\cos x = \sqrt{1 - \sin^2 x}$, such an integral reduces to $\int f(u, \sqrt{1 - u^2})\,du$, where $u = \sin x$. While this need not be rational, it may come under some known head.

In an analogous manner

B. $\int f(\sin x, \cos x)\sin x\,dx = -\int f(\sqrt{1 - u^2}, u)\,du$, where $u = \cos x$.

C. $\int f(\tan x, \sec x)\sec^2 x\,dx = \int f(u, \sqrt{u^2 + 1})\,du$, where $u = \tan x$.

D. $\int f(\cot x, \csc x)\csc^2 x\,dx = -\int f(u, \sqrt{u^2 + 1})\,du$, where $u = \tan x$.

E. $\int f(\tan x, \sec x)\tan x\sec x\,dx = \int f(\sqrt{u^2 - 1}, u)\,du$, where $u = \sec x$.

F. $\int f(\cot x, \csc x)\cot x\csc x\,dx = -\int f(\sqrt{u^2 - 1}, u)\,du$, where $u = \csc x$.

In regard to the integral

G. $\int \sin^m x\cos^n x\,dx,$

four important cases are to be distinguished as readily manageable.

(1) *Either m or n is an odd positive integer*, the other being any number including zero. Thus, to fix the ideas, if $m = 2r + 1$.

$$\int \sin^{2r+1} x\cos^n x\,dx = \int (1 - \cos^2 x)^r\cos^n x\sin x\,dx.$$

Letting $\cos x = u$, this reduces to a sum of integrals of the type $\int u^s du$.

(2) *If neither m nor n is an odd positive integer, but one of them is an odd negative integer*, the same method applies. Thus if

$$m = -(2r-1), \int \frac{\cos^n x\,dx}{\sin^{2r-1} x} = \int \frac{\cos^n x \sin x\,dx}{\sin^{2r} x} = -\int \frac{u^n du}{(1-u^2)^r},$$

where $u = \cos x$.

(3) *$m + n$ is an even negative integer.* Then

$$\int \sin^m x \cos^n x\,dx = \int \frac{\sin^m x}{\cos^m x} \cos^{m+n} x\,dx = \int \tan^m x \sec^{-(m+n)} x\,dx.$$

Since $-(m+n)$ is an even positive integer, this comes under C above.

(4) *m and n are both even positive integers.* Here use is made of the trigonometric relations

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x),\ \cos^2 x = \tfrac{1}{2}(1+\cos 2x),\ \sin x \cos x = \tfrac{1}{2}\sin 2x,$$
$$\cos ax \cos bx = \tfrac{1}{2}[\cos(a+b)x + \cos(a-b)x],$$

enabling one to reduce the integral to a sum of known forms involving the sines or cosines of integral multiples of x. Thus consider

$$\int \sin^4 x \cos^2 x\,dx.$$

$$\begin{aligned}\sin^4 x \cos^2 x &= \sin^2 x \sin^2 x \cos^2 x = \tfrac{1}{4}\sin^2 x \sin^2 2x\\ &= \tfrac{1}{16}(1-\cos 2x)(1-\cos 4x)\\ &= \tfrac{1}{16}(1-\cos 2x - \cos 4x + \cos 2x \cos 4x)\\ &= \tfrac{1}{16}(1-\cos 2x-\cos 4x) + \tfrac{1}{32}(\cos 6x + \cos 2x).\end{aligned}$$

$$\therefore \int \sin^4 x \cos^2 x\,dx = \frac{x}{16} - \frac{\sin 2x}{64} - \frac{\sin 4x}{64} + \frac{\sin 6x}{192} + c.$$

An alternative method for this exercise is

$$\sin^4 x \cos^2 x = \sin^2 x \sin^2 x \cos^2 x = \tfrac{1}{8}(1 - \cos 2x)\sin^2 2x$$

$$= \tfrac{1}{16}(1 - \cos 4x) - \tfrac{1}{8}\sin^2 2x \cos 2x.$$

$$\therefore \int \sin^4 x \cos^2 x\, dx = \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + c.$$

H. $\int \sin mx \sin nx\, dx.$ Using the trigonometric formula

$$\sin A \sin B = \tfrac{1}{2}\cos(A - B) - \tfrac{1}{2}\cos(A + B),$$

$$\int \sin mx \sin nx\, dx = \tfrac{1}{2}\int \cos(m - n)x\, dx - \tfrac{1}{2}\int \cos(m + n)x\, dx$$

$$= \frac{\sin(m - n)x}{2(m - n)} - \frac{\sin(m + n)x}{2(m + n)} + c.$$

I. $\int \sin mx \cos nx\, dx.$ Using the trigonometric formula

$$\sin A \cos B = \tfrac{1}{2}\sin(A + B) + \tfrac{1}{2}\sin(A - B),$$

$$\int \sin mx \cos nx\, dx = \tfrac{1}{2}\int \sin(m + n)x\, dx + \tfrac{1}{2}\int \sin(m - n)x\, dx$$

$$= -\frac{\cos(m + n)x}{2(m + n)} - \frac{\cos(m - n)x}{2(m - n)} + c.$$

J. $\int \cos mx \cos nx\, dx.$ Using the trigonometric formula

$$\cos A \cos B = \tfrac{1}{2}\cos(A + B) + \tfrac{1}{2}\cos(A - B)$$

$$\int \cos mx \cos nx\, dx = \tfrac{1}{2}\int \cos(m + n)x\, dx + \tfrac{1}{2}\int \cos(m - n)x\, dx$$

$$= \frac{\sin(m + n)x}{2(m + n)} + \frac{\sin(m - n)x}{2(m + n)} + c.$$

When $m = n$, the integrals *H*, *I*, *J*, reduce to forms already treated in this section.

Exercises on the methods of this section will be found at the end of the next section.

59. Reduction formulas. At times it is not practicable to evaluate a given integral by a single step. Integration by parts or other processes often enables one to make the value of the integral depend upon that of a simpler one and so obtain a formula whose repeated application ultimately gives the value of the integral. Such a formula is known as a *reduction formula.* In Case 4 of § 53 we made use of a formula of this kind. It was, replacing z by x and β by a,

$$\alpha.\quad \int\frac{dx}{(x^2+a^2)^n}$$
$$=\frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}}+\frac{2n-3}{2(n-1)a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}.$$

In order to familiarize the student with the utility of reduction formulas several of them will be deduced here for convenient reference. It is not advised that the student memorize them.

Integrating by parts, we have

$$\int\sin^m x\,dx=\int\sin^{m-1}x\sin x\,dx$$
$$=-\sin^{m-1}x\cos x+(m-1)\int\sin^{m-2}x\cos^2 x\,dx$$
$$=-\sin^{m-1}x\cos x+(m-1)\int\sin^{m-2}x\,dx-(m-1)\int\sin^m x\,dx.$$

Hence

$$\beta.\quad \int\sin^m x\,dx=-\frac{1}{m}\sin^{m-1}x\cos x+\frac{m-1}{m}\int\sin^{m-2}x\,dx.$$

Repeated use of this formula leads finally to the integration of a constant factor times $\int dx$ or $\int\sin x\,dx$; *i.e.* to the constant factor times x or $-\cos x$. Similarly

$$\gamma.\quad \int\cos^n x\,dx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x\,dx.$$

δ. $\int \sin^m x \cos^n x \, dx$

$$= -\frac{1}{m+n}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx.$$

ε. $\int \sin^m x \cos^n x \, dx$

$$= \frac{1}{m+n}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx.$$

Thus, for example, using ε,

$$\int \sin^4 x \cos^2 x \, dx = \tfrac{1}{6}\sin^5 x \cos x + \tfrac{1}{6}\int \sin^4 x \, dx.$$

Using β twice

$$\int \sin^4 x \, dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x \, dx$$

$$= -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x.$$

$$\int \sin^4 x \cos^2 x \, dx$$

$$= \tfrac{1}{6}\sin^5 x \cos x - \tfrac{1}{24}\sin^3 x \cos x - \tfrac{1}{16}\sin x \cos x + \frac{x}{16} + c.$$

As an exercise, it is suggested that the student deduce the formula α, by letting $x = \alpha \tan \theta$ and using γ.

A reduction formula for $\int \tan^m x \, dx$ is obtained by making use of the fact that $\tan^2 x = \sec^2 x - 1$. Thus if $m \geqq 2$,

$$\int \tan^m x \, dx = \int \tan^{m-2} x \sec^2 x \, dx - \int \tan^{m-2} x \, dx.$$

Hence

θ. $$\int \tan^m x \, dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx.$$

Similarly

η. $$\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx.$$

EXERCISES

Evaluate the following:

1. $\displaystyle\int \frac{\sin^5 x}{\sqrt{\cos x}}\,dx.$

2. $\displaystyle\int \tan^3 x \sec^3 x\,dx.$

3. $\displaystyle\int \tan^4 x \sec^4 x\,dx.$

4. $\displaystyle\int \cot^3 x \csc^4 x\,dx.$

5. $\displaystyle\int \cot^4 x\,dx.$

6. $\displaystyle\int \tan^5 x\,dx.$

7. $\displaystyle\int (\tan x + \cot x)^3\,dx.$

8. $\displaystyle\int \frac{\cos x\,dx}{1 + \sin^2 x}.$

9. $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{1 - \tan^2 x}}.$

10. $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{1 - \tan x}}.$

11. $\displaystyle\int \frac{\sin x\,dx}{a + b\cos x}.$

12. $\displaystyle\int \frac{\cot x \csc x\,dx}{\sqrt{1 + \csc^2 x}}.$

13. $\displaystyle\int \frac{\sin^3 x\,dx}{1 + \cos x}.$

14. $\displaystyle\int \frac{dx}{\sin x \cos^3 x}.$

15. $\displaystyle\int \frac{\cos^2 x\,dx}{\sin^4 x}.$

16. $\displaystyle\int \sec^3 x \csc x\,dx.$

17. $\displaystyle\int (\tan x - 1)^2\,dx.$

18. $\displaystyle\int \frac{(\cos x - \sin x)^2}{\sin x}\,dx.$

19. $\displaystyle\int \frac{\sin^2 x}{\cos^3 x}\,dx.$

20. $\displaystyle\int \frac{\sin x\,(3\cos x + 5)\,dx}{\sin^2 x \cos x + 6}.$

21. $\displaystyle\int \sin^4 x\,dx.$

22. $\displaystyle\int \sin^2 x \cos^6 x\,dx.$

23. $\displaystyle\int \sin 2x \cos 3x\,dx.$

24. $\displaystyle\int \sin 3x \sin x\,dx.$

25. $\displaystyle\int \sin 5x \cos 2x\,dx.$

In the following exercises m and n are integers.

26. $\displaystyle\int_{-\pi}^{\pi} \sin^2 mx\,dx.$

27. $\displaystyle\int_{-\pi}^{\pi} \cos^2 nx\,dx.$

28. $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx\,dx.$

29. $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx\,(m \neq n).$

30. $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\,dx\,(m \neq n).$

60. Wallis' formulas. The numerical values of the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^m x\, dx, \quad \int_0^{\frac{\pi}{2}} \cos^n x\, dx, \quad \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$$

for all possible positive integral values of m and n greater than 1 are given so readily by certain formulas, known as Wallis' formulas, it seems proper to introduce these here.

1. $\int_0^{\frac{\pi}{2}} \sin^m x\, dx.$

If $m = 1$, $\int_0^{\frac{\pi}{2}} \sin x\, dx = -\cos x\Big]_0^{\frac{\pi}{2}} = 1.$

If $m > 1$, we have, using the reduction formula β of § 59,

$$\int_0^{\frac{\pi}{2}} \sin^m x\, dx = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} \sin^{m-2} x\, dx. \qquad (1)$$

This expresses the value of the desired integral in terms of a similar one in which the exponent of $\sin x$ has been reduced by 2. Applying this successively, reducing the exponent of $\sin x$ by 2 each time, we finally arrive at a multiple of either the integral

$$\int_0^{\frac{\pi}{2}} \sin x\, dx = -\cos x\Big]_0^{\frac{\pi}{2}} = 1 \quad \text{if } m \text{ is odd,}$$

or of

$$\int_0^{\frac{\pi}{2}} dx = x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \quad \text{if } m \text{ is even.}$$

That is, using (1) repeatedly,

$$\int_0^{\frac{\pi}{2}} \sin^m x\,dx = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} \sin^{m-2} x\,dx$$

$$= \frac{(m-1)(m-3)}{m(m-2)}\int_0^{\frac{\pi}{2}} \sin^{m-4} x\,dx = \cdots$$

$$= \frac{(m-1)(m-3)\cdots 4\cdot 2}{m(m-2)\cdots 5\cdot 3}\int_0^{\frac{\pi}{2}} \sin x\,dx \quad \text{if } m \text{ is odd}$$

$$= \frac{(m-1)(m-3)\cdots 3\cdot 1}{m(m-2)\cdots 4\cdot 2}\int_0^{\frac{\pi}{2}} dx \qquad \text{if } m \text{ is even.}$$

Hence we have the first pair of the Wallis' formulas

V. $\int_0^{\frac{\pi}{2}} \sin^m x\,dx$

$$= \begin{cases} \dfrac{(m-1)(m-3)\cdots 4\cdot 2}{m(m-2)\cdots 5\cdot 3} & \text{if } m \text{ is odd and } > 1, \\[2ex] \dfrac{(m-1)(m-3)\cdots 3\cdot 1}{m(m-2)\cdots 4\cdot 2}\dfrac{\pi}{2} & \text{if } m \text{ is even.} \end{cases}$$

For the first of these formulas let $m = 2p+1$. On multiplying numerator and denominator by the numerator, the value of the integral is found to be equal to $\dfrac{2^{2p}(p!)^2}{(2p+1)!}$.

For the second of these formulas let $m = 2p$. On multiplying numerator and denominator by the denominator, the value of the integral is found to be equal to $\dfrac{(2p)!}{2^{2p}(p!)^2}$.

2. $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin^n\left(\frac{\pi}{2} - x\right)dx, \text{ since } \cos x = \sin\left(\frac{\pi}{2} - x\right).$$

Letting $\frac{\pi}{2} - x = u$, $dx = -du$. Moreover, when $x = \begin{Bmatrix} 0 \\ \frac{\pi}{2} \end{Bmatrix}$, $u = \begin{Bmatrix} \frac{\pi}{2} \\ 0 \end{Bmatrix}$; that is, when $x = 0$, $u = \frac{\pi}{2}$; when $x = \frac{\pi}{2}$, $u = 0$.

$$\therefore \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n\left(\frac{\pi}{2} - x\right) dx = -\int_{\frac{\pi}{2}}^{0} \sin^n u\, du$$

$$= \int_0^{\frac{\pi}{2}} \sin^n u\, du.$$

Since the value of a definite integral does not depend upon the letter used as the variable, we see that

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

Hence Wallis' formulas **V** also hold for $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$.

3. $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$.

If $m = 1$, $\int^{\frac{\pi}{2}} \sin x \cos^n x\, dx = \left. \frac{-\cos^{n+1} x}{n+1} \right]_0^{\frac{\pi}{2}} = \frac{1}{n+1}$.

If $n = 1$, $\int_0^{\frac{\pi}{2}} \sin^m x \cos x\, dx = \left. \frac{\sin^{m+1} x}{m+1} \right]_0^{\frac{\pi}{2}} = \frac{1}{m+1}$.

If both $m, n > 1$, we may use either reduction formula δ or ϵ of § 59. Using ϵ, we have

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x\, dx. \qquad (2)$$

This expresses the value of the desired integral in terms of a similar one in which the exponent of cos x has been reduced by 2. Applying this successively, we finally arrive at a multiple of the integral

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}\Bigg]_0^{\frac{\pi}{2}} = \frac{1}{m+1} \text{ if } n \text{ is odd,}$$

or of

$$\int_0^{\frac{\pi}{2}} \sin^m x \, dx = \left\{ \begin{array}{l} 1 \text{ if } m = 1 \\ \text{given by } \mathbf{V} \text{ if } m > 1 \end{array} \right\} \text{ if } n \text{ is even.}$$

That is, using (2) repeatedly,

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{(n-1)(n-3)}{(m+n)(m+n-2)}\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-4} x \, dx$$

$$= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= \frac{(n-1)(n-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+5)(m+3)}\int^{\frac{\pi}{2}} \sin^m x \cos x \, dx \quad \text{if } n \text{ is odd}$$

$$= \frac{(n-1)(n-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots(m+4)(m+2)}\int_0^{\frac{\pi}{2}} \sin^m x \, dx \quad \text{if } n \text{ is even.}$$

Hence

$$\textbf{W.} \quad \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

$$= \left\{ \begin{array}{l} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+5)(m+3)(m+1)} \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is odd and } > 1 \\ \dfrac{(n-1)(n-3)\ 3\cdot 1\cdots(m-1)(m-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+4)(m+2)m(m-2)5\cdots 3} \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is even, } m \text{ is odd and } > 1, \\ \dfrac{(n-1)(n-3)\cdots 3\cdot 1\cdot(m-1)(m-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots(m+4)(m+2)m(m-2)\cdots 4\cdot 2}\dfrac{\pi}{2} \\ \qquad\qquad\qquad\qquad \text{if both are even.} \end{array} \right.$$

Since $(m+n)(m+n-2)\cdots(m+4)(m+2)m(m-2)\cdots 5\cdot 3$ may be written

$$(n+m)(n+m-2)\cdots(n+4)(n+2)n(n-2)\cdots 5\cdot 3,$$

the right-hand member of the second formula of **W** may also be put into the form $\dfrac{(m-1)(m-3)\cdots 4\cdot 2}{(n+m)(n+m-2)\cdots(n+3)(n+1)}$, entirely symmetrical to that of the first formula of **W**, as should be expected.

We have thus established Wallis' formulas, which, while not worth memorizing, will be found serviceable in practice. For convenient reference they are tabulated below.

$$\textbf{V.}\quad \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 5\cdot 3} & \text{if } n \text{ is odd and } > 1, \\[2ex] \dfrac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\,\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

$$\textbf{W.}\quad \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+5)(m+3)(m+1)} \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is odd and } > 1, \\[2ex] \dfrac{(m-1)(m-3)\cdots 4\cdot 2}{(n+m)(n+m-2)\cdots(n+5)(n+3)(n+1)} \\ \qquad\qquad\qquad\qquad \text{if } m \text{ is odd and } > 1, \\[2ex] \dfrac{(m-1)(m-3)\cdots 3\cdot 1\cdot(n-1)(n-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots 4\cdot 2}\,\dfrac{\pi}{2} \\ \qquad\qquad\qquad\qquad \text{if both are even.} \end{cases}$$

Ex. 1. $\displaystyle\int_0^{\frac{\pi}{2}} \cos^7 x\,dx = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3} = \frac{16}{35}.$

Ex. 2. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{10} x\,dx = \frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{63\,\pi}{512}.$

Ex. 3. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x \cos 2\,x\,dx = \int_0^{\frac{\pi}{2}} \sin^3 x(\cos^2 x - \sin^2 x)dx$

$$= \int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x\,dx - \int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \frac{2}{5\cdot 3} - \frac{4\cdot 2}{5\cdot 3} = -\frac{2}{5}.$$

Ex. 4. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x\,dx = \frac{3\cdot 1\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{3\,\pi}{512}.$

Ex. 5. $\displaystyle\int_0^{a} x^6\sqrt{a^2 - x^2}\,dx.$ Let $x = a\sin\theta$. Then

$$dx = a\cos\theta\,d\theta.$$

When $x = \left\{\begin{matrix}0\\a\end{matrix}\right\},\quad \theta = \left\{\begin{matrix}0\\ \frac{\pi}{2}\end{matrix}\right\}.$

$$\therefore \int_0^a x\sqrt[6]{a^2 - x^2}\,dx = a^8\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\,d\theta$$

$$= a^8\,\frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{5\,a^8\pi}{256}.$$

Ex. 6. $\displaystyle\int_0^{a} x(a^{\frac{1}{3}} - x^{\frac{1}{3}})^{\frac{3}{2}}\,dx.$ Let $x = a\sin^6\theta$. Then

$$dx = 6\,a\sin^5\theta\cos\theta\,d\theta,\quad (a^{\frac{1}{3}} - x^{\frac{1}{3}})^{\frac{3}{2}} = a^{\frac{1}{2}}\cos^3\theta.$$

When $x = \left\{\begin{matrix}0\\a\end{matrix}\right\},\quad \theta = \left\{\begin{matrix}0\\ \frac{\pi}{2}\end{matrix}\right\}.$

$$\therefore \int_0^a x(a^{\frac{1}{3}} - x^{\frac{1}{3}})^{\frac{3}{2}}\,dx = 6\,a^{\frac{5}{2}}\int_0^{\frac{\pi}{2}} \sin^{11}\theta\cos^4\theta\,d\theta$$

$$= 6\,a^{\frac{5}{2}}\,\frac{10\cdot 8\cdot 6\cdot 4\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5} = \frac{512\,a^{\frac{5}{2}}}{15015}.$$

Remark. — At times it is possible to select a new variable so as to bring a given definite integral into the form evaluated by one of Wallis' formulas. Thus consider

Ex. 7. $\int_0^{\pi}(1-\cos x)^4\,dx$. Noting that, if $x = 2u$, then when

$$x = \begin{Bmatrix}\pi \\ 0\end{Bmatrix},\quad u = \begin{Bmatrix}\frac{\pi}{2} \\ 0\end{Bmatrix};\ \text{also } 1-\cos x = 2\sin^2\frac{x}{2} = 2\sin^2 u.$$

$$\therefore \int_0^{\pi}(1-\cos x)^4\,dx = 32\int_0^{\frac{\pi}{2}}\sin^8 u\,du = 32\,\frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\,\frac{\pi}{2} = \frac{35\,\pi}{8}.$$

Ex. 8. $\int_0^a\sqrt{(2\,ax-x^2)^3}\,dx = \int_0^a\sqrt{[a^2-(x-a)^2]^3}\,dx$

$$= a^4\int_{-\frac{\pi}{2}}^{0}\cos^4\theta\,d\theta,\ \text{when } x-a = a\sin\theta.$$

Putting $\theta = u + \frac{\pi}{2}$, the limits of the resulting integral become 0 and $\frac{\pi}{2}$ respectively; moreover

$$\cos^4\theta = \cos^4\left(u+\frac{\pi}{2}\right) = \sin^4 u.$$

$$\therefore \int_0^a\sqrt{[2\,ax-x^2]^3}\,dx = a^4\int_0^{\frac{\pi}{2}}\sin^4 u\,du = a^4\,\frac{3\cdot1}{4\cdot2}\,\frac{\pi}{2} = \frac{3\,\pi a^4}{16}.$$

EXERCISES

Evaluate the following:

1. $\int_0^{\frac{\pi}{2}}\sin^5 x\,dx.$
2. $\int_0^{\frac{\pi}{2}}\cos^3 x\sin^2 2\,x\,dx.$
3. $\int_0^{\frac{\pi}{2}}\cos^3 2\,x\,dx.$
4. $\int_0^{\frac{\pi}{2}}\sin^6 x\cos^4 x\,dx.$
5. $\int_0^{a}x^2(a^2-x^2)^{\frac{3}{2}}\,dx.$
6. $\int_0^{a}x(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\,dx.$
7. $\int_0^{\pi}\sin^2 x\cos^4 x\,dx.$
8. $\int_0^{\frac{\pi}{4}}\sin^2 x\cos^4 x\,dx.$

61. Summary. At the outset, it should be emphasized that the various methods of this chapter are not mutually exclusive. Many functions can be integrated by each of several methods, and experience will frequently enable one to select the simplest one among these. No general rule can be given. But a knowledge of what possible methods are available in any case will prove a great aid. The general facts are the following:

I. It is absolutely essential that the student know the *standard forms* given in § 51; for all methods consist in reducing a given integral to the consideration of one or more of these forms.

II. *Rational fractions* can be decomposed into sums of simpler rational fractions, each of which can be integrated by a known method. See § 53.

III. *Rational functions of the trigonometric functions* reduce to rational functions of the new variable t as the result of the transformation $\tan\frac{x}{2} = t$. See § 57.

IV. *Certain types of functions of the trigonometric functions* are enumerated in §§ 58 and 59, for which more or less obvious methods of reduction to standard forms are given. Some of these come under the next heading.

V. *Introduction of new variable,* the general purpose of which is to reduce the integral either to a standard form or to a form for which some general method of evaluation is known. The following cases may be noted:

1. Since, if u is any function of x and u' is $\frac{du}{dx}$,

$$\int f(u)\, u'\, dx = \int f(u)\, du,$$

the integral will usually take a simpler form when u is taken as a new variable. This method was first introduced in § 36, and has been used consistently throughout the present chapter.

2. Algebraic functions which are not rational can frequently be transformed into rational functions of a new variable. The following cases were given in § 54:

A. For $f(x,\ (a+bx)^{\frac{p}{q}},\ (a+bx)^{\frac{r}{s}}, \cdots)$, including the case where $a=0$, $b=1$, let $a+bx=u^k$, where k is the least common denominator of the several fractional powers of $a+bx$.

B. For $f(x,\ \sqrt{ax^2+bx+c})$

if $a>0$, let $\sqrt{ax^2+bx+c}=u-x\sqrt{a}$

if $a<0$, $c>0$, let $\sqrt{ax^2+bx+c}=ux-\sqrt{c}$; in this case or in general

if $a<0$ and $\sqrt{ax^2+bx+c}=\sqrt{-a}\sqrt{(x-\alpha)(\beta-x)}$, where α and β are real, let $\sqrt{\dfrac{x-\alpha}{\beta-x}}=u$.

The student should be on the alert for a simplifying transformation, such as $u=\dfrac{1}{x}$ mentioned in the Remark of § 54.

C. For $\int x^m(a+bx^n)^{\frac{r}{s}}\,dx$

if $\dfrac{m+1}{n}$ is an integer including zero, let $a+bx^n=u^s$,

if $\dfrac{m+1}{n}+\dfrac{r}{s}$ is an integer, including zero, let $a+bx^n=x^nu^s$.

3. Algebraic functions involving special types of radicals can be transformed into rational functions of the trigonometric functions. See § 55. Thus

for $f(x,\ \sqrt{a^2-x^2})$ let $x=a\sin\theta$ or $a\cos\theta$,

for $f(x,\ \sqrt{a^2+x^2})$ let $x=a\tan\theta$ or $a\cot\theta$,

for $f(x,\ \sqrt{x^2-a^2})$ let $x=a\sec\theta$ or $a\csc\theta$.

VI. *Integration by parts.* This powerful method for reducing a given integrand to another of more promising form is usually our last resort, when other known methods fail. But in practice it is advisable to bear it in mind and apply it whenever the other methods, even when known to work, are apt to require considerable labor. It is often a time saver.

In general before actually beginning to integrate a function, it is advisable to examine it to see what possible methods are applicable, so as to select the most promising one, in case several are available, and also with a view of possibly simplifying the form by making use of trigonometric or algebraic reductions.

EXERCISES

Evaluate the following:

1. $\int\left(x+\frac{1}{x}\right)^3 dx.$

2. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3 dx.$

3. $\int\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2 dx.$

4. $\int\frac{x\,dx}{x^2+1}; \quad \int\frac{x^2+1}{x}\,dx.$

5. $\int\frac{x^2+1}{(x-1)^3}\,dx.$

6. $\int\frac{3x-2}{2x-3}\,dx.$

7. $\int\frac{4\,dx}{x^3+4x}.$

8. $\int(a^2-x^2)^{\frac{3}{2}}\,dx;$

   $\int x(a^2-x^2)^{\frac{3}{2}}\,dx.$

9. $\int(a^2+x^2)^{\frac{1}{2}}\,dx;$

   $\int x(a^2+x^2)^{\frac{1}{2}}\,dx.$

10. $\int\frac{x^2\,dx}{4-x^3}; \quad \int\frac{x^3\,dx}{4-x^2}.$

11. $\int\sqrt{2ax}\,dx.$

12. $\int x\sin^2 x\,dx.$

13. $\int x\sin 2x\,dx.$

14. $\int\frac{dx}{\cos^2 3x}.$

15. $\int e^{\cot x}\csc^2 x\,dx.$

16. $\int(\cos 2x-2\cos^2 x)\,dx.$

17. $\int(\tan x+\cot x)\,dx.$

18. $\int\frac{dx}{\sin^2 x}; \quad \int\frac{\cos x\,dx}{\sin^2 x}.$

19. $\int(\tan x-1)^2\,dx.$

20. $\int\frac{\sqrt{1+\log x}}{x}\,dx.$

21. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx.$

22. $\int \frac{\cos \sqrt{x}\,dx}{\sqrt{x}}.$

23. $\int \frac{(\cos x - \sin x)^2}{\cos x}\,dx.$

24. $\int \frac{dx}{(x^2+4)^2}$ [let $x=2\tan\theta$].

25. $\int \frac{dx}{(x^2-4)^2}$ [let $x=2\sec\theta$.]

26. $\int \frac{dx}{x\log x}$; $\int x\log x\,dx$; $\int \frac{\log x\,dx}{x}.$

27. $\int \frac{x\,dx}{a^4-x^4}$; $\int \frac{x\,dx}{\sqrt{a^4-x^4}}$; $\int \frac{x\,dx}{\sqrt{x^4-a^4}}.$

28. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$; $\int \frac{e^{2x}+1}{e^{2x}-1}\,dx.$

29. $\int \frac{e^x\,dx}{1+e^{2x}}$; $\int \frac{dx}{e^x+e^{-x}}.$

30. $\int \frac{e^x\,dx}{1+e^x}$; $\int \frac{dx}{1+e^{-x}}.$

31. $\int e^{\sin x}\cos x\,dx.$

32. $\int x^2 e^{-x}\,dx.$

33. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx.$

34. $\int \left(\frac{e^x}{x} - \frac{2\,e^x}{x^3}\right)dx.$

35. $\int e^{x^2}(2\,x^2+1)\,dx.$

36. $\int xe^{x^2}\,dx$; $\int x^3e^{x^2}\,dx.$

37. $\int \frac{xe^x}{(1+x)^2}\,dx.$

38. $\int \frac{dx}{x^2+6\,x+13}$; $\int \frac{x\,dx}{x^2+6\,x+13}$; $\int \frac{x^2\,dx}{x^2+6\,x+13}.$

39. $\int \frac{dx}{\sqrt{x^2+6\,x+13}}$; $\int \frac{x\,dx}{\sqrt{x^2+6\,x+13}}.$

40. $\int \frac{dx}{\sqrt{2+x-x^2}}$; $\int \frac{x\,dx}{\sqrt{2+x-x^2}}.$

41. $\int \frac{dx}{x\sqrt{2-x-x^2}}$; $\int \frac{dx}{x^2\sqrt{2-x-x^2}}.$

42. $\int \frac{dx}{x\sqrt{1-x^2}}.$

43. $\int \sqrt{\frac{x+1}{x-1}}\,dx$; $\int \sqrt{\frac{x+1}{x-2}}\,dx.$

44. $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}.$

45. $\int \sqrt{x^2 - a^2}\,dx$;

$\int x\sqrt{x^2 - a^2}\,dx$.

46. $\int \sqrt{x^2 + a^2}\,dx$;

$\int x\sqrt{x^2 + a^2}\,dx$.

47. $\int \sqrt{a^2 - x^2}\,dx$;

$\int x\sqrt{a^2 - x^2}\,dx$.

48. $\int \frac{dx}{x^2(1 + x^3)^{\frac{5}{3}}}$.

49. $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}}$; $\int \frac{x\,dx}{(a^2 - x^2)^{\frac{3}{2}}}$;

$\int \frac{dx}{x(a^2 - x^2)^{\frac{3}{2}}}$.

50. $\int x\sqrt{x + a}\,dx$.

51. $\int \frac{dx}{x^2\sqrt{x^2 + a^2}}$.

52. $\int \frac{dx}{(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{3}{2}}}$.

53. $\int \frac{dx}{(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{3}{2}}}$.

54. $\int \frac{(x - 1)^{\frac{1}{3}}}{x}\,dx$.

55. $\int \frac{x - 1}{x^{\frac{1}{3}}}\,dx$.

56. $\int \frac{x^{\frac{1}{2}} + x^{\frac{1}{3}}}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}\,dx$.

57. $\int \frac{dx}{\sqrt{x + 4} - \sqrt{x - 4}}$.

58. $\int \frac{dx}{\sqrt{4 + x} - \sqrt{4 - x}}$.

59. $\int (1 - x^{\frac{2}{3}})^{\frac{3}{2}}\,dx$.

60. $\int \frac{x^3\,dx}{\sqrt{1 + x^2}}$.

61. $\int \frac{dx}{x^5 + x^4}$.

62. $\int \frac{x\,dx}{x - 1}$.

63. $\int \frac{dx}{(x^2 - 1)^2}$; $\int \frac{x\,dx}{(x^2 - 1)^2}$.

64. $\int \frac{x^3 + 1}{x^2 + 1}\,dx$.

65. $\int \frac{4x^2 + 8x + 1}{4x^3 - x}\,dx$.

66. $\int \frac{x^2 - 2}{x^3 - x^2}\,dx$.

67. $\int \frac{2x^2 + 4x + 1}{x^3 + 2x^2 + x}\,dx$.

68. $\int \frac{1 + x}{x + x^3}\,dx$.

69. $\int \frac{x^2 + 4x + 6}{(x + 1)^3}\,dx$.

70. $\int \frac{dx}{1 + \sqrt{x}}$.

71. $\int \frac{dx}{(1 - x)\sqrt{x}}$; $\int \frac{dx}{(1 + x)\sqrt{x}}$.

72. $\int \frac{dx}{x - 3\sqrt{x}}$.

73. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx.$

74. $\int \frac{dx}{x(1-x^{\frac{1}{3}})}.$

75. $\int x \cos 2x\,dx.$

76. $\int x \log x\,dx;\ \int x^3 \log x\,dx.$

77. $\int x \tan^{-1} x\,dx.$

78. $\int x \sin^{-1} x\,dx.$

79. $\int \sec^5 x\,dx.$

80. $\int \sec^6 x\,dx.$

81. $\int \sin x \log(1+\sin x)\,dx.$

82. $\int \cot^4 x\,dx.$

83. $\int \cot^3 x\,dx.$

84. $\int \sqrt{\sin x + 1}\,dx.$

85. $\int \sin^3 x \cos^3 x\,dx.$

86. $\int \sin^4 x \cos^2 x\,dx.$

87. $\int \sin^4 x \cos^3 x\,dx.$

88. $\int \frac{\sin^2 x}{\cos^4 x}\,dx.$

89. $\int \frac{\sin^2 x}{\cos^3 x}.$

90. $\int \frac{\sin^3 x}{\cos^2 x}.$

91. $\int \sin x \cos 3x\,dx.$

92. $\int \cos 2x \cos 4x\,dx.$

93. $\int \sin 3x \sin 5x\,dx.$

94. $\int \frac{\sec^3 x - \tan^3 x}{\sec x - \tan x}\,dx.$

95. $\int \sec x \csc^2 x\,dx.$

96. $\int \frac{\sin x\,dx}{2-\sin^2 x}.$

97. $\int \frac{\cos 2x}{\cos^2 x}\,dx.$

98. $\int \frac{dx}{1-\sin x}.$

99. $\int \frac{dx}{1+\cos x};\ \int \frac{\cos x\,dx}{1+\cos x}.$

100. $\int \sec^2 x \tan x\,dx;$

$\int \tan^2 x \sec x\,dx.$

101. $\int \frac{dx}{\sin x + \cos x + 1}.$

62. Tables of integrals. As the evaluation of an integral is usually an incidental problem, a means rather than an end in itself, tables of the integrals apt to arise in the ordinary course of routine work have been compiled to save the time and labor necessary to evaluate these integrals. Such a table will be found in the Appendix of this book.

The training involved in mastering the previous sections of this chapter is of great value for an expert use of the tables, and is advised most strongly. Once skill in applying the various methods of integration has been acquired, the student is advised to use the tables freely.

The arrangement of the integration formulas in the tables is very simple. After a few elementary principles, there are given, in order, the integrals of rational algebraic functions, irrational algebraic functions, and transcendental functions.

Attention should be called to the fact that many reduction formulas appear, by means of which the problem of evaluating the desired integral is replaced by one involving a simpler integral. As a a rule, repeated application of a reduction formula enables one to evaluate the integral. Examples of some of the more important reduction formulas have appeared in this chapter.

To become familiar with the tables of integrals, the student should use them in evaluating the integrals of the previous section.

63. Mechanical integration. Approximate determination of integrals. We have called attention, from time to time, to the fact that integration is not a direct process. No general rule can be given for finding the expression for a function $F(x)$ whose derivative is the prescribed function $f(x)$. As a matter of fact, there is no reason for supposing that the function $F(x)$ can always be expressed in terms of elementary functions when $f(x)$ is given at will. Moreover it is easy to give illustrations of cases where the indefinite integral is not of this type. For example, such is the case for $\int e^{-x^2}dx$, $\int \frac{e^x}{x}dx$, $\int \frac{\sin x}{x}dx$.

Many important definite integrals have been evaluated for certain values of the limits of the integrals even when the indefinite integral is not known in terms of elementary functions. Beyond noting this fact, the scope of this book will not permit our going into it.

Graphical methods have been devised for drawing indefinite integrals.

Also mechanical integrators, such as the planimeter and the integraph, have been constructed to evaluate definite integrals very accurately. They are machines for measuring areas of curves in the plane. The definite integral $\int_a^b f(x)\,dx$ being equal to the area under the curve $y = f(x)$ from the ordinate $x = a$ to $x = b$, the value of the integral is ascertained by drawing the curve accurately, and then, by means of the integrating machine, measuring the area under the curve.

64. Simpson's rules. In Fig. 54 let AB be the curve $y = f(x)$. We shall find a good approximate value for the definite integral $\int_a^b f(x)dx$ by noting that it is equal to the area under the curve from $x = a$ to $x = b$.

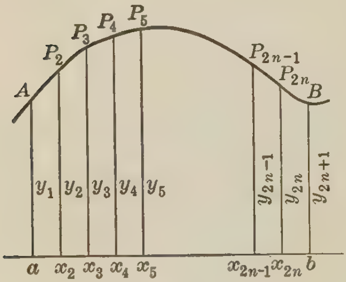

FIG. 54

Divide the interval (a, b) on the x-axis into an even number, say $2n$, of equal subintervals, each $h = \frac{b-a}{2n}$ in length. At each of the points of subdivision of the interval erect the ordinate of the curve. We thus have marked out on the curve $2n+1$ points $A, P_2, P_3, \cdots, P_{2n}, B$, whose coördinates are (a, y_1), (x_2, y_2), (x_3, y_3), $\cdots$, (x_{2n}, y_{2n}), (b, y_{2n+1}) respectively, where $b - x_{2n} = x_{2n} - x_{2n-1} = \cdots = x_2 - a = h$.

It is easy to find the coefficients in the equation of the parabola $y = a_0 + a_1x + a_2x^2$, with vertical axis, that passes through three consecutive points marked out on the curve, and then find the area under that part of the parabola. Thus taking the first three points A, P_2, P_3, the calculation will be simplified if temporarily the ordinate through P_2 is taken as the y-axis. The coördinates of the three points are then $(-h, y_1)$, $(0, y_2)$, (h, y_3) respectively. The parabola $y = a_0 + a_1x + a_2x^2$ will pass through these if

$$a_0 = y_2, \quad a_1 = \frac{y_3 - y_1}{2h}, \quad a_2 = \frac{y_1 + y_3 - 2y_2}{2h^2}.$$

Now

$$\int_{-h}^{h}(a_0 + a_1x + a_2x^2)\,dx = a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3}\Big]_{-h}^{h}$$
$$= 2a_0h + \tfrac{2}{3}a_2h^3.$$

Replacing a_0 and a_2 by their values, we have

Area under parabola from A to $P_3 = \frac{h}{3}(y_1 + 4y_2 + y_3)$.

In an analogous manner,

Area under parabola from P_3 to $P_5 = \frac{h}{3}(y_3 + 4y_4 + y_5)$.

. . . .

Area under parabola from P_{2n-1} to $B = \frac{h}{3}(y_{2n-1} + 4y_{2n} + y_{2n+1})$.

The sum of all these areas equals

$$\alpha. \quad \frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \cdots + 2y_{2n-1} + 4y_{2n} + y_{2n+1}).$$

This area is, as a rule, a good approximation of the desired area, even for relatively small values of n. It is known as

Simpson's One-third Rule. — *Divide the interval* (a, b) *into* $2n$ *equal parts,* h. *Let the corresponding ordinates of the curve be* $y_1, y_2, y_3, \cdots y_{2n+1}$. *Take the sum of the first and last ordinates, twice the sum of the remaining odd ordinates, and four times the sum of all the even ordinates. Multiplying the sum of these three sums by* $\frac{h}{3}$ *gives an approximate value of* $\int f(x)\,dx$.

This is the simplest one of Simpson's rules. A closer approximation can be obtained by dividing the interval (a, b) into $3n$ equal parts, erecting ordinates at the extremities of the subintervals and then replacing the part of the curve joining four consecutive points thus marked out on it by the cubical parabola

$$y = a_0 + a_1x + a_2x^2 + a_3x^3$$

determined by these points. The sum of areas under the n cubical parabolas will be a good approximation of the area under the given curve. This sum turns out to be

$$\beta.\ \tfrac{3}{8}h\,(y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + 3y_8 + \cdots + 3y_{3n} + y_{3n+1}),$$

which is known as *Simpson's three-eighths rule.*

The idea can be extended still further, but the calculations become long and too involved for practical purposes.

Ex. 1. Making use of the fact that $\int_1^3 \frac{dx}{x} = \log 3$, find an approximate value of log 3, using Simpson's first rule. Here $y = \frac{1}{x}.$

For $n = 2$, $y_1 = 1$, $y_2 = \frac{2}{3}$, $y_3 = \frac{1}{2}$, $y_4 = \frac{2}{5}$, $y_5 = \frac{1}{3}$; $h = \frac{1}{2}$.

Simpson's rule gives 1.2667, evidently too large.

For $n = 3$, $y_1 = 1$, $y_2 = \frac{3}{4}$, $y_3 = \frac{3}{5}$, $y_4 = \frac{1}{2}$, $y_5 = \frac{3}{7}$, $y_6 = \frac{3}{8}$, $y_7 = \frac{1}{3}$; $h = \frac{1}{3}$. Simpson's rule gives 1.09894, a good approximation.

For $n = 5$ Simpson's rule gives 1.09866, a still better approximation.

Ex. 2. Making use of the fact that $\int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} 1 = \frac{\pi}{4}$, find an approximate value of π, using Simpson's first rule, and letting $n = 5$.

Ex. 3. Find by Simpson's first rule an approximate value of the area under the hyperbola $xy = 10$ from $x = 1$ to $x = 6$. Let $n = 5$.

CHAPTER VIII

GEOMETRICAL APPLICATIONS

65. Polar coördinates. There is no such simple geometrical interpretation of the derivative of either of the polar coördinates with respect to the other, as there is in the case of the rectangular coördinates. Still there is a simple and important formula connecting the derivative $\frac{d\rho}{d\theta}$ with an angle determining the direction of the tangent line at the point (ρ_0, θ_0). In the figure let

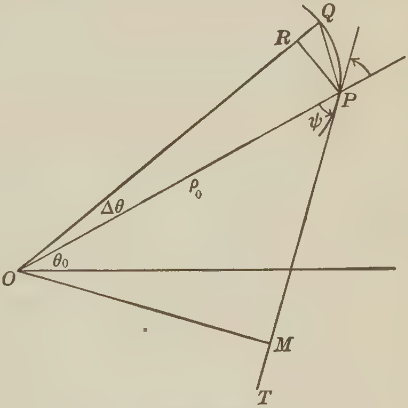

FIG. 55

$O \equiv$ the origin.

$OP \equiv$ the radius vector to the point P on the curve.

$PT \equiv$ the tangent to the curve at P.

$\psi \equiv$ the angle OPT between these two lines measured in the sense indicated by the arrow head; that is, the angle made by the tangent with the radius vector. If the coördinates of P are (ρ_0, θ_0), those of a neighboring point Q are $(\rho_0+\Delta\rho, \theta_0+\Delta\theta)$.

Draw the chord PQ, and PR perpendicular to OQ.

As Q is made to approach P, the angle RQP approaches the angle ψ.

In the right-triangle OPR, $PR = \rho_0 \sin \Delta\theta$ and $OR = \rho_0 \cos \Delta\theta$. Hence $RQ = \rho_0 + \Delta\rho - \rho_0 \cos \Delta\theta$, and

$$\tan RQP = \frac{\rho_0 \sin \Delta\theta}{\rho_0(1 - \cos \Delta\theta) + \Delta\rho} = \frac{\rho_0 \dfrac{\sin \Delta\theta}{\Delta\theta}}{\rho_0 \dfrac{1 - \cos \Delta\theta}{\Delta\theta} + \dfrac{\Delta\rho}{\Delta\theta}}.$$

In § 41, we saw that $\underset{\Delta\theta \to 0}{L} \dfrac{\sin \Delta\theta}{\Delta\theta} = 1.$

We must still find the limit of $\dfrac{1 - \cos \Delta\theta}{\Delta\theta}$ as $\Delta\theta \to 0$.

$$\frac{1 - \cos \Delta\theta}{\Delta\theta} = \frac{2 \sin^2 \dfrac{\Delta\theta}{2}}{\Delta\theta} = \sin \frac{\Delta\theta}{2} \cdot \frac{\sin \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}}.$$

The limit of the first factor is 0, and that of the second is 1. Hence

$$\underset{\Delta\theta \to 0}{L} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0.$$

$$\therefore \underset{\Delta\theta \to 0}{L} \tan RQP = \frac{\rho_0}{(D_\theta \rho)_0};$$

i.e.

$$\tan \psi = \frac{\rho_0}{(D_\theta \rho)_0} = \frac{\rho_0}{\rho'_0} = \rho_0 \left(\frac{d\theta}{d\rho}\right)_0. \tag{1}$$

Here $\left(\dfrac{d\theta}{d\rho}\right)_0$ is the value of $\dfrac{d\theta}{d\rho}$ when $\rho = \rho_0$.

66. Polar tangent; polar normal; polar subtangent; polar subnormal. In Fig. 56 OP is the radius vector to the point P, whose coördinates are (ρ_0, θ_0); QR is drawn perpendicular to OP at O. The following lengths can be obtained readily.

1. $PR \equiv P.T.$, the *polar tangent*, the part of the tangent extending from P to the line QR.

In the right triangle ORP, $PR = OP \sec\psi$. Hence

$$P.T. = \frac{\rho_0\sqrt{\rho_0^2+\rho_0'^2}}{\rho_0'}. \tag{1}$$

2. $OR \equiv P.S.T.$, the *polar subtangent*, the projection of PR on the line QR.

In the same right triangle,

$$OR = OP\tan\psi. \quad \text{Hence}$$

$$P.S.T. = \frac{\rho_0^2}{\rho_0'}. \tag{2}$$

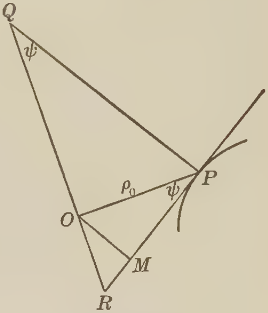

FIG. 56

3. $PQ \equiv P.N.$, the *polar normal*, the part of the normal extending from P to the line QR.

In the right triangle QOP,

$$QP = OP\csc\psi. \quad \text{Hence}$$

$$P.N. = \sqrt{\rho_0^2+\rho_0'^2}. \tag{3}$$

4. $OQ \equiv P.S.N.$, the *polar subnormal*, the projection of PQ on the line QR.

In the same right triangle, $OQ = OP\cot\psi$. Hence

$$P.S.N. = \rho_0'. \tag{4}$$

5. $OM \equiv p_t$, the distance of the tangent from the pole O.

In the right triangle OMP, $OM = OP\sin\psi$. Hence

$$p_t = \frac{\rho_0^2}{\sqrt{\rho_0^2+\rho_0'^2}}. \tag{5}$$

67. Curves in polar coördinates. Among the curves whose equations are simply expressed in polar coördinates, the following may be mentioned:

1. The *lemniscate*

$$\rho^2 = a^2\cos 2\theta.$$

It has the form indicated in Fig. 57. Differentiating, we have

$$2\rho\rho' = -2a^2 \sin 2\theta. \quad \therefore \rho' = -\frac{a^2 \sin 2\theta}{\rho}.$$

$$\text{Hence } \tan\psi = \frac{\rho}{\rho'} = -\frac{\rho^2}{a^2 \sin 2\theta} = -\cot 2\theta.$$

$$\therefore \psi = 2\theta + \frac{\pi}{2} + k\pi, \tag{1}$$

where k is a positive or negative integer, or zero. Formula (1) expresses a characteristic property of the lemniscate. It enables one to draw the tangent to the curve at any point P by drawing OM making an angle 3θ with OA, and drawing a perpendicular to this through P. This perpendicular is tangent to the curve because, when P is in the first quadrant,

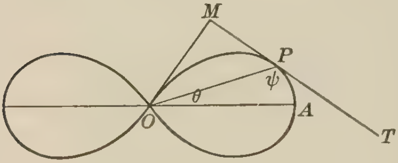

FIG. 57

$$\psi = OPT = POM + OMP = 2\theta + \frac{\pi}{2}.$$

Obvious modifications of the proof must be made when P is not in the first quadrant.

We now have a simple means for finding the points on the curve where the tangents are horizontal (maximum and minimum points). Thus, for the first quadrant, if the tangent is horizontal, the line perpendicular to it makes an angle $\frac{\pi}{2}$ with the horizontal. But this is 3θ. Hence $3\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{6}$.

From this same construction it is also easy to find the distance from the origin to the tangent. From the right triangle OMP, in which $\angle POM = 2\theta$, we have

$$OM = OP \cos POM = \rho \cos 2\theta = \frac{\rho^3}{a^2}.$$

It will be left as an exercise for the student to show that the general expression (5) for OM, deduced in § 66, reduces to this form; also that the polar subtangent and polar subnormal are $\rho \cot 2\theta$ and $\rho \tan 2\theta$ respectively.

2. The *cardioid*

$$\rho = 2a(1 - \cos\theta).$$

Differentiating we have

$$\rho' = 2a\sin\theta.$$

$$\therefore \tan\psi = \frac{\rho}{\rho'} = \frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

Hence $$\psi = \frac{\theta}{2} \quad \text{or} \quad \frac{\theta}{2} + k\pi, \ (k \text{ an integer}). \tag{2}$$

From this it follows that the vectorial angles corresponding to the points P and Q, at which the tangent is vertical, are $\frac{\pi}{3}$ and $-\frac{\pi}{3}$ respectively; for, in the right triangle OPM, we must have

$$\angle OPM = \tfrac{1}{2}\angle POM.$$

Since their sum is $\frac{\pi}{2}$, $\angle POM = \frac{\pi}{3}$.

Similarly the vectorial angles corresponding to the highest and lowest points R and S of the curve (where the tangents are horizontal) are $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$ respectively; for, in the figure $\angle MOR$ and $\angle ORT$ are supplementary; *i.e.* $\theta + \frac{\theta}{2} = \pi$, or $\theta = \frac{2\pi}{3}$.

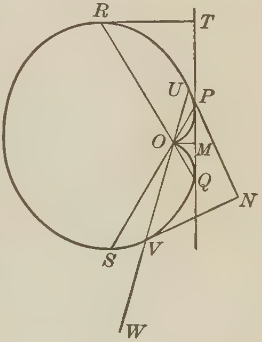

FIG. 58

Two radii corresponding to the vectorial angles θ_0 and $\theta_0 + \pi$ are collinear. They may be said to form a chord through the origin (or pole). Corresponding to these the two values of ψ are $\frac{\theta_0}{2}$ and $\frac{\theta_0}{2}+\frac{\pi}{2}\cdot$ In the figure, let UV be a chord through O. At U, $\psi=\angle OUN=\frac{\theta_0}{2}$; at V, $\psi=\angle WVN=\frac{\theta_0}{2}+\frac{\pi}{2}\cdot$ Hence in the triangle VUN, $\angle VNU=\frac{\pi}{2}$; that is, *the tangents at the extremities of a chord through O are mutually perpendicular.*

It will be left as a very simple exercise to show that all the chords through O have the same length, $4a$.

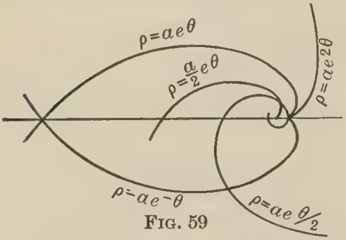

FIG. 59

3. The *logarithmic* or *equiangular spiral*

$$\rho = ae^{b\theta},$$

where a and b are constants.

$$\rho' = abe^{b\theta}. \qquad (3)$$

$$\therefore \tan\psi=\frac{1}{b}, \text{ a constant;}$$

hence the name equiangular. It will be left as an exercise to show that

$$\text{polar subtangent} = \frac{\rho}{b},$$

$$\text{polar subnormal} = b\rho.$$

Also show that the two equiangular spirals $\rho = ae^{\theta}$ and $\rho = ae^{-\theta}$ meet each other at right angles.

4. The *spiral of Archimedes*

$$\rho = a\theta.$$

Here $\rho' = a.$

$$\therefore \tan\psi = \theta. \qquad (4)$$

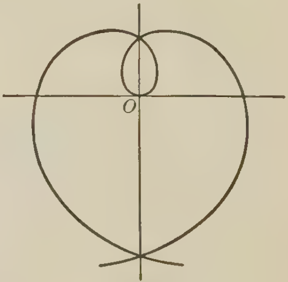

FIG. 60

ψ starts out by being 0 and approaches $\frac{\pi}{2}$ as θ increases, in absolute value, either positively or negatively.

The polar subtangent $= \rho$.

The polar subnormal $= a$.

5. The *hyperbolic spiral*

$$\rho = \frac{a}{\theta}.$$

Here $$\rho' = -\frac{a}{\theta^2}.$$

$$\therefore \tan\psi = -\theta. \qquad (5)$$

The polar subtangent $= \theta$.

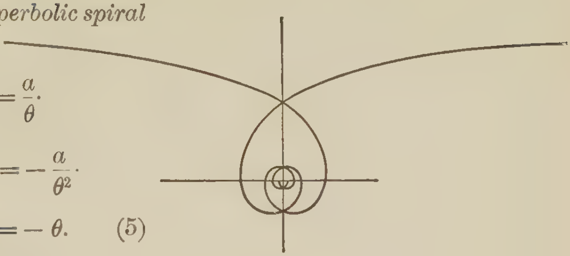

FIG. 61

EXERCISES

Sketch and discuss the following curves.

1. $\rho^2 = a^2 \sin 2\,\theta$.
2. $\rho^2 \sin 2\,\theta = a^2$.
3. $\rho = a \sin\theta$.
4. $\rho^2 = a^2 \sin\theta$.
5. $\rho = a(1 + \sin\theta)$.
6. $\rho = a \sin\theta + b\cos\theta$.
7. $\rho = a - b\cos\theta$.

In Ex. 7 consider separately the cases $a = 0$, $a < b$, $a = b$, $a > b$.

8. $\rho = a \sin 2\,\theta$.
9. $\rho = a \sin\frac{\theta}{2}$.
10. $\rho = a \sin 3\,\theta$.
11. $\rho = a \sin\frac{\theta}{3}$.
12. $\rho = \dfrac{a}{\sin\theta} = a\csc\theta$.
13. $\rho = \dfrac{a}{1 - \sin\theta}$.
14. $\rho = a\sec^2\dfrac{\theta}{2} = \dfrac{2\,a}{1 + \cos\theta}$.
15. $\rho^2 = a^2 \sin 3\,\theta$.
16. $\rho^2 = a^2 \sin n\theta$.
17. Compare the curves $\rho = f(\sin\theta)$ and $\rho = f(\cos\theta)$.

68. Cycloid. An important curve whose equation is most readily handled in parametric form is the cycloid, which is the curve traced by a point on the circumference of a circle rolling, without slipping, along a straight line. In Fig. 62, suppose that P was originally at O. Since the original position was left, the circle has turned through the angle $PCM \equiv t$. If a is the radius of the circle,

$$OM = \text{arc } MP = at.$$

Then $$x = OR = OM - PN = at - a\sin t.$$
$$y = RP = MC - NC = a - a\cos t.$$

The parametric equations of the cycloid are thus

$$x = a(t - \sin t), \quad y = a(1 - \cos t). \tag{1}$$

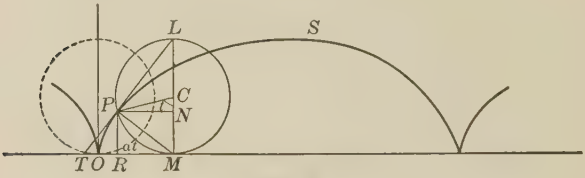

FIG. 62

Differentiating, we have

$$dx = a(1 - \cos t)\,dt, \quad dy = a\sin t\,dt.$$

$$\therefore \frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2};$$

i.e. $\tan\theta = \cot\frac{t}{2}$. Hence

$$\theta = \frac{\pi}{2} - \frac{t}{2} \quad \text{or} \quad \frac{\pi}{2} - \frac{t}{2} + k\pi, \ (k \text{ an integer}).$$

In the figure, $\theta = \angle PTM$.

In the isosceles triangle PCM, $\angle PMC = \frac{\pi}{2} - \frac{t}{2}$. $\therefore \angle PMT = \frac{t}{2}$, and $\angle TPM$ is a right angle. Hence *the normal to the cycloid at P passes through M, the lowest point of the rolling circle.* The student should have little difficulty in showing now that *the tangent to the cycloid at P passes through L, the highest point of the rolling circle.*

It will be left as an exercise also to show that $D_x^2 y = -\frac{a}{y^2}$, which is evidently always negative. What does this mean?

When the origin is at S, the vertex (*i.e.* the highest point on the curve), the equations of the cycloid take the form

$$x = a(t_1 + \sin t_1), \quad y = a(-1 + \cos t_1), \tag{2}$$

where $t_1 = t - \pi$.

Show that the equations of an inverted cycloid (*i.e.* where the circle rolls on the lower side of the line) referred to the vertex as origin are

$$x = a(t + \sin t), \quad y = a(1 - \cos t). \tag{3}$$

If some point attached to the rolling circle, not on the circumference but at a distance $b (\neq a)$ from the center is selected as the tracing point, a curve known as a *trochoid* is obtained. According as $b > a$ or $b < a$, the curve is designated as an *oblate* or a *prolate* trochoid. As illustrations may be mentioned the curves generated by a point on the outer edge of the flange of a railway car wheel and by a point on the spoke of a wheel respectively.

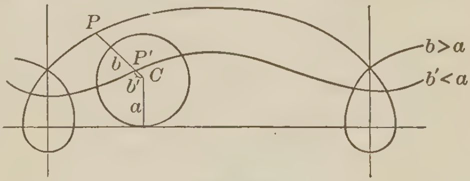

FIG. 63

The cycloid has a series of cusps. These are replaced by a series of loops in the case of the oblate trochoid. The prolate trochoid has neither cusps nor loops. See Fig. 63.

It will be left as an exercise to show that in both cases, the parametric equations of the trochoid are

$$x = at - b \sin t, \quad y = a - b \cos t. \tag{4}$$

Area of an arch of the cycloid. Using the equations in the form (1), we have

$$\text{Area} = \int_0^{2\pi a} y\, dx = a^2 \int (1 - \cos t) d(t - \sin t)$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt.$$

Noting that $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$,

$$\text{Area} = a^2\int_0^{2\pi}(\tfrac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t)\,dt$$

$$= a^2(\tfrac{3}{2}t - 2\sin t + \tfrac{1}{4}\sin 2t)\Big]_0^{2\pi} = 3\pi a^2,$$

three times the area of the generating circle.

Ex. 1. Show that the area of an arch of a prolate trochoid is $2\pi a^2 + \pi b^2$.

Ex. 2. At what point of a prolate trochoid is the curve steepest?

REMARK. — As the circle in Fig. 62 or Fig. 63 rolls without slipping along OM, C moves in a straight line parallel to OM; that is, P revolves about C at a fixed distance from it, while C moves in a straight line. The path of P is a cycloid if P (in its relative motion along a circle about C) and C (actually) travel equal distances; otherwise it is a trochoid, oblate or prolate according as P travels a greater or less distance than C. Thus, a point on the pedal of a bicycle travels in a prolate trochoid.

69. Epicycloid; hypocycloid. When a circle rolls without slipping upon the outside of a fixed circle, any selected point on the circumference of the rolling circle generates an *epicycloid.* If it rolls upon the inside of the other circle, the point generates a *hypocycloid.*

For both curves let a be the radius of the fixed circle, and b that of the rolling circle. When the moving circle has rolled through the angle $\phi \equiv PBC'$, the line AB joining the centers of the circles will make an angle $\theta \equiv CAB$ with its initial position. Since arc $PC' =$ arc CC', $b\phi = a\theta$ or

$$\phi = \frac{a\theta}{b}. \tag{1}$$

We shall take A, the center of the fixed circle, as the origin and AC as the axis of x. If P is the point on the moving circle generating the locus, its coördinates (x, y) in the case of the epicycloid (Fig. 64) are

$$x = AM = AL - PN, \qquad y = MP = LB - NB.$$

Evidently $\qquad AL = (a + b)\cos\theta, \quad LB = (a + b\sin\theta).$

Moreover $\angle BPN = \theta + \phi.$

$$\therefore\ PN = b\cos(\theta + \phi), \quad NB = b\sin(\theta + \phi);$$

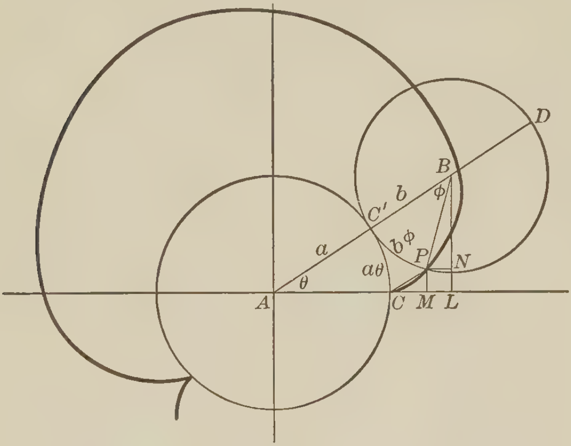

FIG. 64

whence, using (1), we have, finally,

$$\left.\begin{aligned} x &= (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta, \\ y &= (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta, \end{aligned}\right\} \qquad (2)$$

the *equations of the epicycloid in parametric form.*

In the case of the hypocycloid (Fig. 65) the coördinates of P are

$$x = AM = AL + NP, \qquad y = MP = LB - NB.$$

Now $\qquad AB = a - b \quad$ and $\angle BPN = \phi - \theta.$

$$\therefore\ AL = (a - b)\cos\theta, \quad NP = b\cos(\phi - \theta) = b\cos\frac{a-b}{b}\theta.$$

$$LB = (a - b)\sin\theta, \quad NB = b\sin(\phi - \theta) = b\sin\frac{a-b}{b}\theta.$$

Hence *the equations of the hypocycloid in parametric form* are

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta, \\ y &= (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta. \end{aligned}\right\} \tag{3}$$

Algebraically the epicycloid and hypocycloid are the same type of curve. We can pass from the equations of one to those of the other by changing b into $-b$.

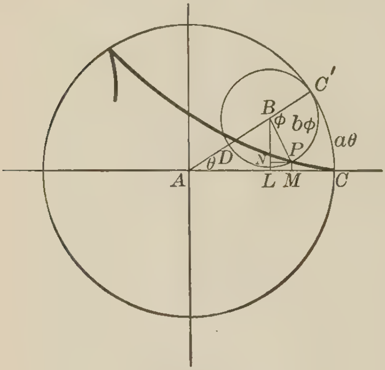

FIG. 65

It will be left as an exercise to prove that in the case of both curves the *tangent at P passes through D and the normal at P passes through C'.*

The following special cases should be noted:

1. If $b = a$, the resulting epicycloid is the cardioid (§ 67, 2)

$$x = 2a\cos\theta - a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta.$$

To identify this curve, we move the origin to the cusp C, whose coördinates are $(a, 0)$; thus

$$\begin{aligned} x_1 = 2a\cos\theta - a\cos 2\theta - a &= 2a\cos\theta - 2a\cos^2\theta \\ &= 2a\cos\theta(1-\cos\theta) \end{aligned}$$

$$\begin{aligned} y_1 = 2a\sin\theta - a\sin 2\theta &= 2a\sin\theta - 2a\sin\theta\cos\theta \\ &= 2a\sin\theta(1-\cos\theta). \end{aligned}$$

Incidentally the equations

$$x = 2a\cos\theta(1-\cos\theta), \quad y = 2a\sin\theta(1-\cos\theta) \tag{4}$$

may be noted as the *parametric equations of the cardioid, when the cusp is taken as the origin.*

Since θ is already used here to represent an angle, we shall use Θ to represent the vectorial angle PCM (Fig. 64).

It is not difficult to show, either by elementary geometry or analytically, that $\Theta = \theta$ when $b = a$. Hence, using this in the first equation of (4) and passing to polar coördinates,

$$\rho \cos \Theta = 2\, a \cos \Theta(1 - \cos \Theta), \text{ or } \rho = 2\, a(1 - \cos \Theta),$$

the polar equation of the cardioid.

2. If $b = \frac{a}{4}$, the resulting hypocycloid is known as the *hypocycloid of four cusps* or *astroid*. From (3) its equations assume the form

$$x = \tfrac{3}{4} a \cos \theta + \tfrac{1}{4} a \cos 3\,\theta = \frac{a}{4}(3 \cos \theta + \cos 3\,\theta) = a \cos^3 \theta,$$

$$y = \tfrac{3}{4} a \sin \theta - \tfrac{1}{4} a \sin 3\theta = \frac{a}{4}(3 \sin \theta - \sin 3\,\theta) = a \sin^3 \theta.$$

The equations

$$x = a \cos^3 \theta, \qquad y = a \sin^3 \theta \quad (5)$$

may be noted as the *parametric equations of the astroid*. Writing them

$$x^{\frac{1}{3}} = a^{\frac{1}{3}} \cos \theta, \quad y^{\frac{1}{3}} = a^{\frac{1}{3}} \sin \theta,$$

on squaring and adding, we have the rectangular equation of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad (6)$$

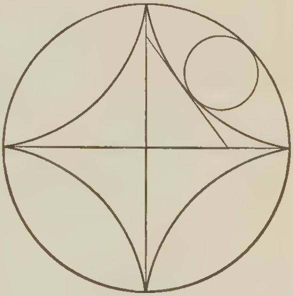

FIG. 66

In Ex. 2, § 22, a characteristic property of the astroid was noted; viz. *the mutually perpendicular diameters joining opposite cusps intercept the same length a for all tangents to the curve.*

3. If $b = \frac{a}{2}$, the resulting hypocycloid of two cusps is nothing but a diameter, traced back and forth.

70. Epitrochoid; hypotrochoid. If instead of taking the tracing point on the circumference of the rolling circle in § 69, some point attached to the rolling circle but at a distance c, different from b, from its center is selected as the tracing point, a curve known as an *epitrochoid* or a *hypotrochoid*, respectively, is generated according as the circle rolls upon the outside or inside of the other circle. The cusps of the epicycloid and hypocycloid are replaced by loops in the epitrochoid and hypotrochoid in the case when $c > b$. When $c < b$, the curve generated has neither cusps nor loops.

As one of the circles in Fig. 64 or Fig. 65 rolls upon the other, its center B moves along a circle with A as center and radius equal to $a + b$ or $a - b$ respectively. Moreover, since there is no slipping, when the rolling circle has revolved about B through an angle ϕ, the line AB has revolved about A through the angle θ, where $a\theta = b\phi$. Thus the curve may be looked upon as generated by P, one end of the line BP, of length β, which revolves about B at the same time that the line AB, of length α, revolves about A, which is fixed. Besides, if the angles through which AB and BP have turned in an interval of time are θ and ϕ respectively,

in the case of the epicycloid $(\alpha - \beta)\,\theta = \beta\phi$, and θ and ϕ have been generated in the same direction,

in the case of the hypocycloid $(\alpha + \beta)\theta = \beta\phi$, and θ and ϕ have been generated in opposite directions.

If the relations connecting θ and ϕ are different from those given, the traced curve is an epitrochoid or hypotrochoid according as AB and BP revolve in the same or opposite directions.

Starting to measure θ and ϕ when A, B, and P are on a straight line with B between A and P, the equations of the generated curve, in all possible cases, are (Fig. 67)

$$x = \alpha \cos \theta + \beta \cos \phi, \quad y = \alpha \sin \theta + \beta \sin \phi. \qquad (1)$$

These are consistent with equations (2) and (3) of § 69, account being taken of the initial values of θ and ϕ, and of the relative algebraic signs of these angles.

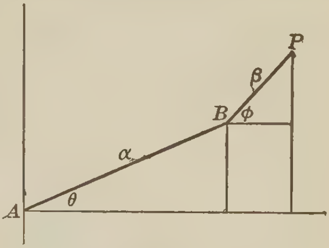

FIG. 67

As a first approximation of the path of a planet about the earth, suppose the earth and planet to travel in circles about the sun, with periods proportional to the three-halves power of their distances from the sun. Relative to the earth, the sun travels about the earth in a circle with the earth's period. With respect to the earth, the path of the planet is an epitrochoid since, using the same notation as before, $\frac{\theta}{\phi} = \frac{\beta^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}}$ and the sun and planet revolve in the same sense. A little consideration will show that the epitrochoid has loops. As a matter of fact, the apparent motion of the planet as seen from the earth is retrogressive while it travels along a loop.

Thus, for example, in the case of Jupiter, the distance of the planet from the sun is approximately 5.2 that of the earth, while its period is about 12 times that of the earth. Taking the distance of the earth from the sun as a unit and noting that $\frac{\theta}{\phi} = \frac{12}{1}$, a first approximation of the path of Jupiter about the earth is given by

$$x = \cos 12\,\phi + 5.2 \cos \phi, \quad y = \sin 12\,\phi + 5.2 \sin \phi.$$

71. Differential of arc in rectangular coördinates. In Fig. 68, let lengths of arc along the curve be measured from some definite point A on it. If P and Q are the points (x, y) and $(x + \Delta x,\ y + \Delta y)$ on the curve, let

$$s = \text{length of arc } AP \qquad \Delta s = \text{length of arc } PQ.$$

Then

$$\frac{\Delta s}{\Delta x}=\frac{\Delta s}{\overline{PQ}}\frac{\overline{PQ}}{\Delta x}=\frac{\Delta s}{\overline{PQ}}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x}=\frac{\Delta s}{\overline{PQ}}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}. \tag{1}$$

To find $\frac{ds}{dx}$ we must determine $\underset{\Delta x \to 0}{L}\frac{\Delta s}{\overline{PQ}}$.

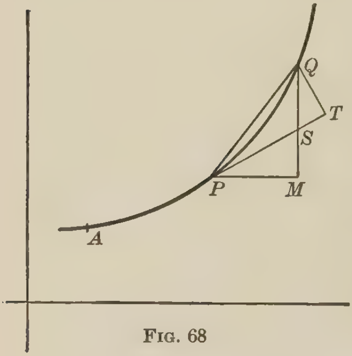

FIG. 68

By taking Δs sufficiently small, we may assume that the arc PQ is always concave to its chord $\overline{PQ}$, and that QT, the perpendicular from Q on the tangent at P, meets the curve at no other point than Q. Then

$$PT+TQ>\Delta s>\overline{PQ}.$$

$$\therefore \frac{PT}{\overline{PQ}}+\frac{TQ}{\overline{PQ}}>\frac{\Delta s}{\overline{PQ}}>1. \tag{2}$$

Designating the angle TPQ by α, $\frac{PT}{\overline{PQ}}=\cos\alpha$ and $\frac{TQ}{\overline{PQ}}=\sin\alpha$.

As Δx approaches zero, α also approaches zero: hence the first term of the inequality (2) approaches 1.

$$\therefore \underset{\Delta x \to 0}{L}\frac{\Delta s}{\overline{PQ}}=1. \tag{3}$$

Passing to the limit for $\Delta x \to 0$, (1) leads to

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \text{ or } ds=\sqrt{dx^2+dy^2}. \tag{4}$$

If the equation of the curve is given in parametric form

$$x=f(t), \quad y=\phi(t),$$

$$dx=f'(t)\,dt, \quad dy=\phi'(t)\,dt.$$

$$\therefore \frac{ds}{dt}=\sqrt{[f'(t)]^2+[\phi'(t)]^2}$$

or

$$ds=\sqrt{f'^2+\phi'^2}\,dt=\sqrt{x'^2+y'^2}\,dt. \tag{5}$$

72. Length of arc. From the fact that

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

we are in a position to find the length of the curve from any point (x_1, y_1) on it to (x_2, y_2) on it, expressed as a definite integral. Thus, if $F(x)$ is an indefinite integral of $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, we have

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = F(x) + c;$$

i.e. the length s_1 from A (Fig. 68) to $(x_1, y_1) = F(x_1) + c$,

the length s_2 from A to $(x_2, y_2) \qquad = F(x_2) + c$.

$\therefore$ the length $s_2 - s_1$, from (x_1, y_1) to $(x_2, y_2) = F(x_2) - F(x_1)$; that is,

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1}$$

The alternative form

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{2}$$

is also available. At times it may be easier to evaluate this integral than the one in (1).

If the equation of the curve is given in parametric form, we have the third form

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3}$$

Formulas (1) and (2) are special cases of (3).

Ex. 1. Find the length of the circumference of the circle $x^2 + y^2 = a^2$.

Here $y = \sqrt{a^2 - x^2}$. $\therefore \dfrac{dy}{dx} = \dfrac{-x}{\sqrt{a^2 - x^2}}$ and

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{a}{\sqrt{a^2 - x^2}}.$$

Because of symmetry, the entire circumference is equal to 4 times the arc in the first quadrant.

$$\therefore\ s = 4\,a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4\,a \sin^{-1}\frac{x}{a}\bigg]_0^a = 4\,a \sin^{-1} 1 = 2\,\pi\,a.$$

Ex. 2. Find the length of an arch of the cycloid.

Any of the forms of the equation given in § 68 may be used. Using (3), the form for the inverted cycloid,

$$x = a(t + \sin t), \qquad y = a(1 - \cos t),$$

$$dx = a(1 + \cos t)dt, \quad dy = a \sin t\, dt,$$

$$\sqrt{dx^2 + dy^2} = a\sqrt{2(1 + \cos t)}dt = 2\,a \cos\frac{t}{2}dt.$$

$$\therefore\ s = \int_{-\pi}^{\pi} \cos\frac{t}{2}\,dt = 4\,a \sin\frac{t}{2}\bigg]_{-\pi}^{\pi} = 8\,a.$$

Note. — If instead of looking for the length of the entire arch, we ask for the length of arc from the vertex to any point on the same arch, we find

$$s = 2\,a\int_0^t \cos\frac{t}{2}dt = 4\,a \sin\frac{t}{2}\bigg]_0^t = 4\,a \sin\frac{t}{2}.$$

$$\therefore\ s^2 = 16\,a^2 \sin^2\frac{t}{2} = 8\,a^2(1 - \cos t) = 8\,ay.$$

This is a characteristic property of the cycloid.

EXERCISES

1. Find the entire length of the hypocycloid of four cusps, using its equation in the form $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; also in the parametric form $x = a \cos^3 t$, $y = a \sin^3 t$.

2. Find the length of a branch of the semicubical parabola $9\,y^2 = 4\,x^3$ from $x = 0$ to $x = 8$.

3. Find the length of the arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $x = 0$ to $x = x_1$.

From this prove that if s_1 is the length of arc from the vertex of the curve to the point (x_1, y_1) on it, $s_1^2 = y_1^2 - a^2$, which is a characteristic property of the catenary.

4. Find the length of the arc of the parabola $x^2 = 4\,ay$ from the vertex to an extremity of its latus rectum.

5. Find the length of the arc of $y = e^x$ from $x = 0$ to $x = \log 7$.

6. Find the length of the cardioid

$$x = 2\,a \cos\theta - a\cos 2\,\theta, \quad y = 2\,a\sin\theta - a \sin 2\,\theta$$

considered as an epicycloid (§ 69, 1).

73. Differential and length of arc in polar coördinates. As before, let length of arc along the curve be measured from some definite point A on it. If P and Q are the points (ρ, θ) and $(\rho + \Delta\rho,\ \theta + \Delta\theta)$ on the curve, let

$s =$ length of arc AP
$\Delta s =$ length of arc PQ.

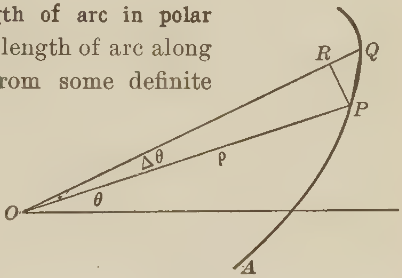

FIG. 69

Then, noting that

$$\text{chord } \overline{PQ} = \sqrt{\overline{PR}^2 + \overline{RQ}^2} = \sqrt{\rho^2 \sin^2\Delta\theta + (\rho + \Delta\rho - \rho\cos\Delta\theta)^2}$$
$$= \sqrt{\rho^2 \sin^2\Delta\theta + [\Delta\rho + \rho(1 - \cos\Delta\theta)]^2},$$

we have

$$\frac{\Delta s}{\Delta\theta} = \frac{\Delta s}{\overline{PQ}}\frac{\overline{PQ}}{\Delta\theta} = \frac{\Delta s}{\overline{PQ}}\sqrt{\rho^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left[\frac{\Delta\rho}{\Delta\theta} + \rho\frac{1 - \cos\Delta\theta}{\Delta\theta}\right]^2}. \qquad (1)$$

As shown in § 71, $\underset{\Delta s \to 0}{L} \dfrac{\Delta s}{\overline{PQ}} = 1$; besides, we have

$$\underset{\Delta\theta \to 0}{L} \frac{\sin\Delta\theta}{\Delta\theta} = 1 \quad (§\ 41), \quad \text{and} \quad \underset{\Delta\theta \to 0}{L} \frac{1 - \cos\Delta\theta}{\Delta\theta} = 0 \quad (§\ 65).$$

Hence, passing to the limit for $\Delta\theta \to 0$, (1) leads to

$$\frac{ds}{d\theta} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2} = \sqrt{\rho^2 + \rho'^2}, \text{ or } ds = \sqrt{\rho^2 d\theta^2 + d\rho^2}. \qquad (2)$$

As an exercise the student should obtain the second form of (2) from the second form of (4), § 71, by passing from rectangular to polar coördinates $x = \rho \cos\theta$, $y = \rho \sin\theta$.

From (2) we have, in a manner entirely analogous to that used in the previous section, for the length of arc from (ρ_1, θ_1) to (ρ_2, θ_2)

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta \tag{3}$$

or

$$s = \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\, d\rho. \tag{4}$$

EXERCISES

1. Find the length of the arc of the logarithmic spiral $\rho = ae^{b\theta}$ from $\theta = 0$ to $\theta = \pi$.

2. Find the length of the entire perimeter of the cardioid $\rho = 2\,a(1 - \cos\theta)$.

3. Find the length of the entire curve $\rho = a \sin^2\frac{\theta}{2}$.

4. Find the length of the arc of the parabola $\rho = a \sec^2\frac{\theta}{2}$ lying to the right of the axis of y.

74. Curvature. As a point moves along a curve the tangent at the point, in general, changes its direction. In Fig. 70, let P be a point on the curve at a distance s measured along the curve from some designated point A. If Q is a neighboring point at a distance $s + \Delta s$ from A, and the inclinations of the tangents at P and Q are α and $\alpha + \Delta\alpha$ respectively, $\frac{\Delta\alpha}{\Delta s}$ is known as the *average curvature* of the arc PQ, and $\frac{d\alpha}{ds}$, its limit as $\Delta s \rightarrow 0$, is the *curvature* of the curve at P.

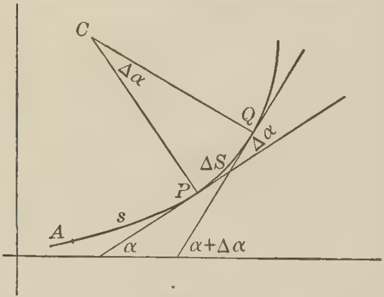

FIG. 70

1. *Rectangular coördinates.* Introducing the variable x,

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dx}\frac{dx}{ds} = \frac{\frac{d\alpha}{dx}}{\frac{ds}{dx}}.$$

Since $\dfrac{dy}{dx} = \tan\alpha$, $\alpha = \tan^{-1}\dfrac{dy}{dx} = \tan^{-1}y'$.

$$\therefore \frac{d\alpha}{dx} = \frac{d\tan^{-1}y'}{dx} = \frac{y''}{1+y'^2}.$$

Moreover, $\dfrac{ds}{dx} = \sqrt{1+y'^2}$. Hence, designating the curvature by K,

$$K = \frac{y''}{(1+y'^2)^{\frac{3}{2}}} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{1}$$

2. *Equation of the curve given in parametric form.*

If $x = f(t)$, $y = \phi(t)$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'}{x'}$.

As before, $\alpha = \tan^{-1}\dfrac{dy}{dx} = \tan^{-1}\dfrac{y'}{x'}$.

$$\therefore \frac{d\alpha}{dt} = \frac{x'y''-y'x''}{x'^2+y'^2}.$$

Introducing the variable t

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dt}\frac{dt}{ds} = \frac{\frac{d\alpha}{dt}}{\frac{ds}{dt}} = \frac{\frac{x'y''-y'x''}{x'^2+y'^2}}{\sqrt{x'^2+y'^2}}.$$

$$\therefore K = \frac{x'y''-y'x''}{(x'^2+y'^2)^{\frac{3}{2}}} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}. \tag{2}$$

3. *Polar coördinates.* Noting that $\alpha = \theta + \psi$ (compare Fig. 71), we have

$$\frac{d\alpha}{d\theta} = 1 + \frac{d\psi}{d\theta}.$$

In § 65 we found $\tan\psi = \dfrac{\rho}{\dfrac{d\rho}{d\theta}} = \dfrac{\rho}{\rho'}$. $\therefore \psi = \tan^{-1}\dfrac{\rho}{\rho'}$, and

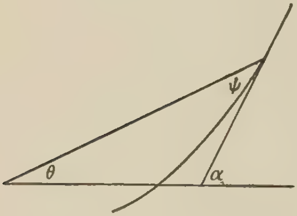

FIG. 71

$$\frac{d\psi}{d\theta} = \frac{\rho'^2 - \rho\rho''}{\rho^2 + \rho'^2}.$$

$$\therefore \frac{d\alpha}{d\theta} = 1 + \frac{\rho'^2 - \rho\rho''}{\rho^2 + \rho'^2}$$

$$= \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{\rho^2 + \rho'^2}.$$

Moreover, $\dfrac{ds}{d\theta} = \sqrt{\rho^2 + \rho'^2}$.

Introducing the variable θ,

$$K = \frac{d\alpha}{ds} = \frac{d\alpha}{d\theta}\frac{d\theta}{ds} = \frac{\dfrac{d\alpha}{d\theta}}{\dfrac{ds}{d\theta}}.$$ Hence

$$K = \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{[\rho^2 + \rho'^2]^{\frac{3}{2}}} = \frac{\rho^2 + 2\left(\dfrac{d\rho}{d\theta}\right)^2 - \rho\dfrac{d^2\rho}{d\theta^2}}{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}. \tag{3}$$

EXERCISES

1. Find the curvature of $2ay = x^2$ at (x, y). Where is it a maximum?

2. Find the curvature of $x^2 - 4y^2 = 16$ at (x, y); at $(4, 0)$; at $(5, -\frac{3}{2})$.

3. Find the curvature of $x = a(t - \sin t)$, $y = a(1 - \cos t)$ at any point of the curve; at the vertex; at a cusp.

4. Find the curvature of $x = a\cos^3 t$, $y = a\sin^3 t$ at any point. Where is it a minimum?

5. Find the curvature of $\rho = 2a(1 - \cos\theta)$. Where is it a minimum?

6. Find the curvature of $\rho = a\theta$.

75. Radius of curvature; center of curvature. In the case of the circle, taking its equations in parametric form,

$$\begin{aligned} x &= a\cos t, & y &= a\sin t, \\ x' &= -a\sin t, & y' &= a\cos t, \\ x'' &= -a\cos t, & y'' &= -a\sin t. \end{aligned}$$

$$\therefore K = \frac{a^2\sin^2 t + a^2\cos^2 t}{(a^2\sin^2 t + a^2\cos^2 t)^{\frac{3}{2}}} = \frac{1}{a}.$$

As might have been expected, the curvature of a circle is the same at all points. But an additional fact of interest is, that the curvature is numerically equal to the reciprocal of the radius of the circle.

This suggests the introduction of the idea of *circle of curvature* at a point P of a curve, by which is meant the circle tangent to the curve at P on the concave side of the curve and having the same curvature as the curve at P.

The radius of the circle of curvature at P is called the *radius of curvature* of the curve at P. Hence *the radius of curvature at any point is the reciprocal of the curvature at that point.*

If R is the radius of curvature, we have

$$R = \frac{1}{K} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \text{or} \quad \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}$$

$$\text{or} \quad \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}, \qquad (1)$$

depending upon the system of coördinates used.

The center of the circle of curvature at P is called the *center of curvature* of the curve at P. To find its coördinates, we note that it lies on the normal at P directed toward the concave side of the curve and at a distance R from P.

Let the coördinates of P be (x_0, y_0) and those of the center of curvature C be $(\bar{x}_0, \bar{y}_0)$. Noting that $\tan\alpha = y'_0$, we have

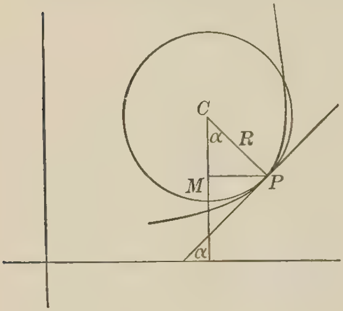

FIG. 72

$$\sin\alpha = \frac{y'_0}{\sqrt{1+y'^2_0}},$$

$$\cos\alpha = \frac{1}{\sqrt{1+y'^2_0}}.$$

Evidently, in Fig. 72,

$$\angle MCP = \alpha,$$

when α is an acute angle.

$$\therefore\ MP = R\sin\alpha = \frac{Ry'_0}{\sqrt{1+y'^2_0}},\quad MC = R\cos\alpha = \frac{R}{\sqrt{1+y'^2_0}}.$$

Replacing R by its first value in (1),

$$MP = \frac{y'_0(1+y'^2_0)}{y''_0} \text{ and } MC = \frac{1+y'^2_0}{y''_0}.$$

These are free of radicals. Hence corresponding to the point P on the curve, no matter what be the algebraic signs of y'_0 and y''_0,

$MP = x_0 - \bar{x}_0$; for C lies to the left of P when y'_0 and y''_0 have the same sign, and C lies to the right of P when y'_0 and y''_0 have opposite signs;

$MC = \bar{y}_0 - y_0$; for C lies above P when y''_0 is positive, and C lies below P when y''_0 is negative. Hence the coördinates of the center of curvature are given by the formulas

$$\bar{x}_0 = x_0 - \frac{y'_0(1+y'^2_0)}{y''_0},\quad \bar{y}_0 = y_0 + \frac{1+y'^2_0}{y''_0}. \tag{2}$$

Ex. Find the radius and center of curvature of $xy = a^2$ at the point (x_0, y_0). In particular find the equation of the circle of curvature of $xy = 4$ at the point $(1, 4)$.

Finding the derivative of y as an implicit function,

$$y' = -\frac{y}{x}. \quad \therefore y'' = \frac{-xy' + y}{x^2} = \frac{2y}{x^2} = \frac{2xy}{x^3} = \frac{2a^2}{x^3}.$$

$$\therefore R = \frac{\left(1 + \frac{y^2}{x^2}\right)^{\frac{3}{2}}}{\frac{2a^2}{x^3}} = \frac{(x^2 + y^2)^{\frac{3}{2}}}{2a^2}$$

For the point (x_0, y_0) we have

$$R = \frac{(x_0{}^2 + y_0{}^2)^{\frac{3}{2}}}{2a^2} = \frac{(x_0{}^2 + y_0{}^2)^{\frac{3}{2}}}{2x_0y_0}.$$

From (2), we have for the center of curvature, using the value $y_0'' = \dfrac{2y_0}{x_0}$,

$$\bar{x}_0 = \frac{3x_0{}^2 + y_0{}^2}{2x_0}, \quad \bar{y}_0 = \frac{x_0{}^2 + 3y_0{}^2}{2y_0}.$$

For the hyperbola $xy = 4$ at the point $(1, 4)$

$$\bar{x}_0 = \tfrac{19}{2}, \quad \bar{y}_0 = \tfrac{49}{8}.$$

The equation of the circle having $(\frac{19}{2}, \frac{49}{8})$ for center and passing through $(1, 4)$ is $x^2 + y^2 - 19x - \frac{49}{4}y + 51 = 0$.

To find the points of intersection of this circle with $xy = 4$, we solve the latter for y, obtaining $y = \dfrac{4}{x}$, and substitute this in the equation of the circle, obtaining after clearing of fractions

$$x^4 - 19x^3 + 51x^2 - 49x + 16 = 0. \tag{3}$$

This rather formidable-looking equation has the root $x = 1$, which is obvious, since the sum of the coefficients equals zero. Hence $x - 1$ is a factor of the left-hand member. Factoring, we find the remaining factor also contains $x-1$ as a factor. Repeating the process, we find that the left-hand member of (3) actually contains the factor $x-1$ three times. As a matter of fact (3) may be written

$$(x-1)^3(x-16)=0.$$

Hence for the points of intersection of circle of curvature and the original curve

$$x=1,\ 1,\ 1,\ 16.$$
$$\therefore y=4,\ 4,\ 4,\ \tfrac{1}{4}.$$

That is, of the four possible points of intersection, three are found to be located at the point (1, 4). This is not an accident. It is always true that *the point of the curve counts, in general, as three of the possible points of intersection of its circle of curvature with the curve.* The circle of curvature at a point of a curve is sometimes referred to as the *osculating circle* of the curve at that point.

We have already called attention to the fact that at a flex the curvature is zero. The osculating circle in the case of a flex is the flex tangent. Correspondingly *the flex point counts,* in general, *as three of the possible intersections of its tangent with the curve.* In the case of an ordinary tangent, the point of contact counts as two of the possible intersections of the tangent with the curve.

It will be left as an exercise for the student to show that, corresponding to the point (2, 2) on $xy=4$, the equation of the osculating circle is

$$x^2+y^2-8x-8y+24=0,$$

and that the point (2, 2) counts as four points of intersection of this circle with the hyperbola.

A sketch of the osculating circles of $xy=4$ at (1, 4) and (2, 2) will show that the former crosses the hyperbola at (1, 4) while the latter lies entirely on one side of the hyperbola.

EXERCISES

1. Find the center of curvature of $6y = x^2$ at $(4, \frac{8}{3})$.

2. Find the radius of curvature of $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{\frac{-x}{a}})$ at (x, y). What is it at the vertex $(0, a)$?

3. Find the radius of curvature of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at (x, y). Compare this with the length of the perpendicular from the origin to the tangent at (x, y).

4. Find the curvature of $ay^2 = x^3$ at (x, y).

5. Find the curvature of $ay = x^3$ at (x, y).

6. Find the radius of curvature of $y = \log \sec x$ at (x, y).

7. Find the radius of curvature of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2c^{\frac{1}{2}}$ at (x, y).

8. Find the radius of curvature of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x, y). What is it at $(a, 0)$ and at $(0, b)$?

9. Find the radius of curvature of $y = e^x$ at (x, y). Where is it a minimum?

10. Show that the radius of curvature of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ at any point is twice the length of the normal at that point.

11. Find the radius of curvature of the epicycloid

$$x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta$$

at any point.

Find the radius of curvature of the following curves at the point (ρ, θ):

12. $\rho = e^{a\theta}$.

13. $\rho = a\sin^3\frac{\theta}{3}$.

14. $\rho = a\sec^2\frac{\theta}{2}$.

15. $\rho^2\cos 2\theta = a^2$.

16. $\rho^2 = a^2\sin 2\theta$.

CHAPTER IX

OTHER APPLICATIONS

76. Simple harmonic motion. A particle is free to move in a straight line subject to an attracting force which is always directed toward a fixed point O on the line of motion and is proportional to the distance of the particle from O. If, at the instant $t = 0$, the particle is at $x = x_0$ and its velocity $v = 0$, determine the motion of the particle.

Let the line of motion be the axis of x and O the origin. If x denotes the position of the particle at any instant,

$$\frac{dx}{dt} = v, \text{ the velocity at that instant,}$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = a, \text{ the acceleration at that instant.}$$

If m is the mass of the particle, we have, from elementary mechanics, force $= ma$. Hence the given law of force may be expressed

$$ma = -k^2x,$$

where the factor $-k^2$ indicates that the force is directed negatively when x is positive, and positively when x is negative, as it should be, since the force is an attractive one.

Since m is constant throughout the motion, we may divide by it. Replacing the negative constant $\frac{-k^2}{m}$ by $-\mu^2$ and noting that $a = \frac{d^2x}{dt^2}$, we have, for the differential equation of motion,

$$\frac{d^2x}{dt^2} = -\mu^2x. \tag{1}$$

Our problem consists in finding x as a function of t which satisfies the equation (1), subject to the conditions that $x = x_0$ and $v = 0$ when $t = 0$.

Recalling that $\frac{d^2x}{dt^2} = \frac{dv}{dt}$, equation (1) may be replaced by

$$\frac{dv}{dt} = -\mu^2 x, \tag{2}$$

which is simpler, in so far as it involves a derivative of the first order instead of the second, but it is more complicated in that it involves three variables x, v, t. But if we take x as the independent variable, we have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v.$$

Our differential equation then takes the form

$$v\frac{dv}{dx} = -\mu^2 x, \tag{3}$$

which involves two variables only. Multiplying by dx, we have the equation

$$v\,dv = -\mu^2 x\,dx, \tag{4}$$

in which the variables are said to be *separated*.[1]

Integrating each member, after having multiplied both by 2,

$$v^2 = -\mu^2 x^2 + c. \tag{5}$$

The constant of integration can be evaluated, recalling that $v = 0$ when $x = x_0$. Using this fact in (5),

$$0 = -\mu^2 x_0^2 + c; \quad i.e.\ c = \mu^2 x_0^2.$$

$$\therefore v^2 = \mu^2(x_0^2 - x^2). \tag{6}$$

---

[1] In general, when a differential equation, involving two variables, has the form

$$M(x)dx + N(x)dy = 0,$$

where M is a function of x only and N is a function of y only, the variables are said to be *separated*. The solution of such an equation is obtained by integrating each term, and equating their sum to a constant of integration.

This determines the velocity at each point of the path. Solving for v, we have $v = \mu\sqrt{x_0^2 - x^2}$.

$$\therefore \frac{dx}{dt} = \mu\sqrt{x_0^2 - x^2}. \qquad (7)$$

Separating the variables,

$$\frac{dx}{\sqrt{x_0^2 - x^2}} = \mu\, dt.$$

Integrating, we have

$$\sin^{-1}\frac{x}{x_0} = \mu t + c. \qquad (8)$$

To evaluate c, we recall that $x = x_0$ when $t = 0$. Hence (8) tells us that

$$\sin^{-1} 1 = c; \ \textit{i.e.}\ c = \frac{\pi}{2}.$$

$$\therefore \sin^{-1}\frac{x}{x_0} = \mu t + \frac{\pi}{2}. \qquad (9)$$

Taking the sine of each side,

$$\frac{x}{x_0} = \sin\left(\mu t + \frac{\pi}{2}\right) = \cos \mu t.$$

$$\therefore x = x_0 \cos \mu t. \qquad (10)$$

Equation (10) enables one to tell all about the motion. Differentiating it, we have

$$\frac{dx}{dt} = v = -x_0 \mu \sin \mu t, \qquad (11)$$

which gives the velocity of the particle as a function of the time, while (6) gives the velocity as a function of the position of the particle. [Show that equations (6), (10), and (11) are consistent, by obtaining (6) from (10) and (11).]

Using (10) and (11), we see that when $t = 0$, x has its maximum value x_0 and $v = 0$. As t grows v becomes negative and x diminishes. When $t = \frac{\pi}{2\mu}$, v has its minimum value and

$x = 0$. From then until $t = \frac{\pi}{\mu}$, v continues to grow algebraically and x is negative; when $t = \frac{\pi}{\mu}$, $v = 0$ and $x = -x_0$. After that v is positive, acquiring its maximum value when $t = \frac{3\pi}{2\mu}$, x in the meanwhile growing algebraically and arriving at 0 when $t = \frac{3\pi}{2\mu}$. From then until $t = \frac{2\pi}{\mu}$, v diminishes ending by being zero, while x grows, arriving at x_0 at the end. As t grows, the process is repeated in precisely the order described. The motion of the particle is backward and forward over the strip from $x = x_0$ to $x = -x_0$, an indefinite number of times. Motion of the sort described is said to be *simple harmonic.* By the *period* of the motion is meant the time that has elapsed between two consecutive passages of the particle over the same point in its path in the same direction. In the present case the period is $T = \frac{2\pi}{\mu}$.

The values v_0 and x_0 respectively which v and x assume at a given instant t_0 are known as the *initial conditions* of the problem. The constants of integration arising in the course of solving the differential equation of the motion have their values determined by these initial conditions.

Ex. 1. If when $t = 0$, $v = v_0$ and $x = 0$, show that

$$x = \frac{v_0}{\mu} \sin \mu t.$$

Ex. 2. In the general case, where $v = v_0$ and $x = x_0$ when $t = t_0$, find the equation of motion.

As before, starting with

$$v\,dv = -\mu^2 x\,dx, \tag{4}$$

we have

$$v^2 = -\mu^2 x^2 + c, \tag{5}$$

Since c is equal to the sum of two squares and is, therefore, positive, let it be μ^2A^2. Then

$$v^2 = \mu^2(A^2 - x^2), \text{ or}$$

$$v = \frac{dx}{dt} = \mu\sqrt{A^2 - x^2}.$$

Separating the variables,

$$\frac{dx}{\sqrt{A^2 - x^2}} = \mu\, dt.$$

Integrating,

$$\sin^{-1}\frac{x}{A} = \mu t + \phi.$$

$$\therefore x = A\sin(\mu t + \phi). \tag{12}$$

Here A and ϕ are the constants of integration. They can be determined by the initial conditions thus:

From (12) and the result of differentiating it,

$$v = A\mu\cos(\mu t + \phi), \tag{13}$$

we have

$$x_0 = A\sin(\mu t_0 + \phi) \quad\text{and}\quad \frac{v_0}{\mu} = A\cos(\mu t_0 + \phi). \tag{14}$$

Squaring both of these and adding, we have

$$x_0^2 + \frac{v_0^2}{\mu^2} = A^2 \quad\text{or}\quad A = \frac{\sqrt{\mu^2 x_0^2 + v_0^2}}{\mu}. \tag{15}$$

Taking the quotients of corresponding members of (14),

$$\frac{\mu x_0}{v_0} = \tan(\mu t_0 + \phi). \quad \therefore \mu t_0 + \phi = \tan^{-1}\left(\frac{\mu x_0}{v_0}\right), \text{ or}$$

$$\phi = \tan^{-1}\left(\frac{\mu x_0}{v_0}\right) - \mu t_0. \tag{16}$$

The values of A and ϕ in terms of t_0, v_0, x_0, given in (15) and (16), are not of great importance. But the physical significance of A and ϕ should be emphasized.

It is easy to show, and will be left as an exercise, that A is the maximum value assumed by x throughout the motion. A is known as the *amplitude*.

Upon the value of ϕ will depend the time when x reaches any point. ϕ is known as the *phase.*

Equation (12) may be taken as the general equation of simple harmonic motion. It will be left as a simple exercise to show that in this case also the period $T = \frac{2\pi}{\mu}$.

It should be noted that the period depends upon the force acting in any particular case, and is independent of the initial conditions, while A and ϕ do depend upon the initial conditions.

Ex. 3. Show that by taking $\phi = \psi + \frac{\pi}{2}$ for the constant of integration in (12), the general equation of simple harmonic motion assumes the form

$$x = A\cos(\mu t + A). \tag{17}$$

Ex. 4. Noting that

$$A\cos\mu t + B\sin\mu t$$
$$= \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\cos\mu t + \frac{B}{\sqrt{A^2 + B^2}}\sin\mu t\right),$$

show that $x = A\cos\mu t + B\sin\mu t$ may be made to take form (12) or (17), and thus also defines simple harmonic motion. What is the amplitude in this case?

Ex. 5. If the force is a repulsive one, the differential equation of motion is

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} = \mu^2 x.$$

If $v = 0$ and $x = x_0$ when $t = 0$, show that

$$v = \mu\sqrt{x^2 - x_0^2} \quad \text{and} \quad x = \frac{x_0}{2}(e^{\mu t} + e^{-\mu t}) = x_0\cosh\mu t.$$

Whence $$v = \frac{x_0\mu}{2}(e^{\mu t} - e^{-\mu t}) = x_0\mu\sinh\mu t.$$

Of course, the motion in this case is not simple harmonic. What is its nature?

77. Uniform circular motion. If the motion of a particle is given by the equations

$$x = r\cos\omega t, \quad y = r\sin\omega t, \tag{1}$$

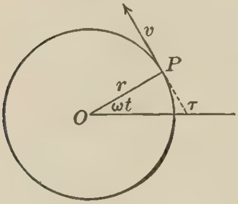

FIG. 73

where r and ω are constants, the path is the circle

$$x^2 + y^2 = r^2,$$

as may be ascertained by eliminating t between equations (1).

For the velocity we have (§ 25)

$$v_x = x' = -r\omega\sin\omega t, \quad v_y = y' = r\omega\cos\omega t.$$

$$\therefore \text{speed} \equiv |v| = \sqrt{v_x^2 + v_y^2} = r\omega, \text{ a constant.} \tag{2}$$

If τ is the angle of inclination of the velocity at any instant,

$$\cos\tau = \frac{v_x}{|v|} = -\sin\omega t, \quad \sin\tau = \frac{v_y}{|v|} = \cos\omega t.$$

$$\therefore \tau = \omega t + \frac{\pi}{2}. \tag{3}$$

The motion is at right angles to the line joining the moving point with the origin, as it must be since the path is a circle.

Motion in a circle with constant speed is known as *uniform circular motion.* Equations (1) are the equations of such motion.

For the acceleration we have (§ 26)

$$a_x = x'' = -r\omega^2\cos\omega t, \quad a_y = y'' = -r\omega^2\sin\omega t.$$

Using formulas (5) and (6) of § 26, viz.,

$$\text{tangential acceleration} \equiv a_t = \frac{x'x'' + y'y''}{\sqrt{x'^2 + y'^2}},$$

$$\text{normal acceleration} \equiv a_n = \frac{x'y'' - y'x''}{\sqrt{x'^2 + y'^2}},$$

we find

$$a_t = 0, \text{ as it should be since the speed is constant,} \tag{4}$$

$$a_n = r\omega^2. \tag{5}$$

The entire acceleration, in this case, is in the direction of the normal.

Using the fact that $|v| = r\omega$, we see that

$$a_n = \frac{|v|^2}{r}. \tag{6}$$

This interesting fact is a special case of what can be said in general of the normal acceleration for a body moving along a curve. Thus, in § 75 we saw that the radius of curvature of a curve $x = f(t)$, $y = \phi(t)$ at any point t is

$$R = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''}.$$

Comparing this with the expression for a_n, we see that

$$a_n = \frac{x'^2 + y'^2}{R} = \frac{|v|^2}{R}; \tag{7}$$

i.e. *the normal acceleration at any point along the path of a particle is equal to the square of the speed divided by the radius of curvature of the path at that point.*

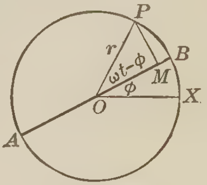

FIG. 74

An interesting fact in connection with uniform circular motion is that the projection of the instantaneous position of the particle upon any diameter of the path oscillates back and forth along this diameter with simple harmonic motion. For, if AB is a diameter making an angle ϕ with OX, the distance from O to M, the projection of P upon AB, is

$$OM \equiv s = r\cos(\omega t - \phi).$$

Ex. 1. Find the path, speed, tangential, and normal acceleration of a particle whose equations of motion are

$$x = a\cos\omega t, \quad y = b\sin\omega t.$$

If $a > b > 0$, at what points do the speed and normal acceleration assume maximum and minimum values?

Ex. 2. Find the equation of the path of a particle whose equations of motion are

$$x = a\cos\omega t, \quad y = a\sin 2\,\omega t.$$

Sketch it. At what points is the speed a maximum or a minimum?

Ex. 3. A particle moves with constant speed around the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Discuss the tangential and normal components of the acceleration.

Ex. 4. If a wheel rolls along a straight line with constant angular velocity ω, find the vertical and horizontal components of the velocity of a point of the rim of the wheel, as it traces the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. [Here θ is used as the parameter, t being reserved for the time. Then $\dfrac{d\theta}{dt} = \omega$.] What is the speed? Where is this a maximum?

78. Simple pendulum. Simple harmonic motion is approximately the motion of many objects in nature, such as the swing of a simple pendulum or of an undamped galvanometer needle through a small arc, the oscillation of a point on the prong of a tuning fork, or the vibration up and down of a weighted particle suspended by an elastic cord or coiled spring. We shall consider the case of the simple pendulum.

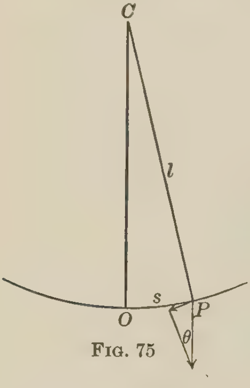

FIG. 75

Let P be a particle of mass m suspended by a massless thread, l in length, from a point C so that it will oscillate freely in a vacuum without friction in a vertical plane. P moves along a circle. If s is the length of arc measured from the lowest point O to P, $s = l\theta$, where θ is the angle which CP makes with CO. Therefore the acceleration in the direction of the tangent to the path at P is $\frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$.

The only force acting is gravity. It is directed vertically downward and is equal to $-mg$ in amount. Its component along the direction of the tangent at P is $-mg\sin\theta$. Hence the equation of motion is

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{1}$$

If θ remains small throughout the motion, we may replace $\sin\theta$ by θ as a first approximation. The equation of motion then takes the form

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta, \tag{2}$$

which has the form, (1) § 76, of the differential equation of simple harmonic motion.

The solution of this equation gives θ as a function of t. But since we are supposing all values of θ throughout the motion to be very small, we are interested in s, the displacement of the particle, rather than in the angle θ; for, if l is large, s may be appreciable while θ is small. Multiplying both members of (2) by l and remembering that $l\theta = s$, our differential equation of motion is

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s. \tag{3}$$

If we start the pendulum swinging by pulling it aside a distance s_0, and then, without imparting to it any initial velocity, we let go and allow it to swing, we have the initial conditions $s = s_0$, $\frac{ds}{dt} = v = 0$ when $t = 0$.

The student should show that, with these initial conditions,

$$s = s_0 \cos\sqrt{\frac{g}{l}}t.$$

On the other hand, if the pendulum is started by giving it a sharp blow when at rest in the position CO, causing it to have an initial velocity v_0, the student should show that the equation of motion is

$$s = v_0\sqrt{\frac{l}{g}}\sin\sqrt{\frac{g}{l}}\,t;$$

also that, no matter what be the initial conditions, the period of the pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$.

79. Motion of body attracted by the earth at a distance. Only for relatively small distances above the earth's surface, compared with R, the radius of the earth, is the acceleration due to gravity even approximately the constant $-g$. For a body at a distance s from the center of the earth where $s > R$, the acceleration due to the earth's attraction is $-\frac{k^2}{s^2}$, where $-k^2$ is the factor of proportionality. To determine it, we note that, when $s = R$, the acceleration is $-g$; *i.e.* $-\frac{k^2}{R^2} = -g$. Whence $-k^2 = -gR^2$. The differential equation of motion is then

$$\frac{d^2s}{dt^2} = -\frac{gR^2}{s^2}. \tag{1}$$

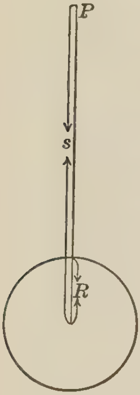

FIG. 76

Here the resistance of the air and the existence of any attracting body, except the earth, are ignored. The earth is assumed to be a sphere.

We shall study the case where the motion is in the direction of the line joining the initial position of the body with the center of the earth. As was done in § 76, we note that

$\frac{ds}{dt}=v$, $\frac{d^2s}{dt^2}=\frac{dv}{dt}$, and then introduce s as the independent variable, thus

$$\frac{d^2s}{dt^2}=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds}.$$

Equation (1) may then be written

$$v\frac{dv}{ds}=-\frac{gR^2}{s^2},$$

or, separating the variables,

$$v\,dv=-gR^2\frac{ds}{s^2}. \tag{2}$$

Multiplying the members by 2 and integrating,

$$v^2=\frac{2\,gR^2}{s}+c. \tag{3}$$

If the body starts at a distance s_0 from the center of the earth and with zero velocity, we have

$$0=\frac{2\,gR^2}{s_0}+c,\quad\text{or}\quad c=-\frac{2\,gR^2}{s_0}.\quad\therefore v^2=2\,gR^2\left(\frac{1}{s}-\frac{1}{s_0}\right),\text{ or}$$

$$v=-R\sqrt{2\,g}\sqrt{\frac{1}{s}-\frac{1}{s_0}}. \tag{4}$$

This determines the velocity of the attracted body at any point of its path. Here the negative sign is used on the right-hand side because the body is falling and we are measuring distances positively upward.

In particular if $s=R$ and $s_0=R+h$, the body reaches the earth's surface with the velocity

$$v=-\sqrt{2\,gh}\sqrt{\frac{R}{R+h}}. \tag{5}$$

Comparing this with (9), § 39, we see that the latter formula is approximately true only for such distances h above the earth's surface for which $\frac{R}{R+h}$ may be taken as unity.

To find the time at which the body arrives at any distance s, we write (4) in the form

$$\frac{ds}{dt} = -R\sqrt{2g}\sqrt{\frac{s_0 - s}{s_0 s}}, \text{ or}$$

$$\sqrt{\frac{s}{s_0 - s}}\,ds = -R\sqrt{\frac{2g}{s_0}}\,dt. \tag{6}$$

To integrate the left-hand member let $s = s_0 \sin^2 u$. Then $ds = 2\,s_0 \sin u \cos u\,du$ and $\sqrt{s_0 - s} = \sqrt{s_0}\cos u$.

$$\therefore \int\sqrt{\frac{s}{s_0 - s}}\,ds = 2\,s_0\int \sin^2 u\,du = s_0\int(1 - \cos 2\,u)du$$

$$= s_0 u - \frac{s_0 \sin 2\,u}{2} = s_0 \sin^{-1}\sqrt{\frac{s}{s_0}} - \sqrt{s_0 s - s^2}.$$

Hence, integrating both members of (6) and equating results,

$$s_0 \sin^{-1}\sqrt{\frac{s}{s_0}} - \sqrt{s_0 s - s^2} = -R\sqrt{\frac{2g}{s_0}}\,t + c.$$

When $t = 0$, $s = s_0$. $\therefore s_0 \sin^{-1} 1 = s_0 \frac{\pi}{2} = c$.

$$\therefore R\sqrt{\frac{2g}{s_0}}\,t = \sqrt{s_0 s - s^2} + s_0\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{s}{s_0}}\right), \text{ or}$$

$$R\sqrt{\frac{2g}{s_0}}\,t = \sqrt{s_0 s - s^2} + s_0 \cos^{-1}\sqrt{\frac{s}{s_0}}. \tag{7}$$

Ex. 1. Show that it will take about 47.5 minutes for a body starting with zero initial velocity from a distance of 4000 miles above the earth's surface, to reach the earth. The radius of the earth is here taken as 4000 miles.

If, instead of starting with zero initial velocity, it starts with velocity v_0, the value of the constant of integration in (3) is $v_0^2 - \frac{2gR^2}{s_0}$. Hence the velocity at any point s along the path is given by

$$v^2 - v_0^2 = 2\,gR^2\left(\frac{1}{s} - \frac{1}{s_0}\right). \tag{8}$$

If we ask, what is the least velocity with which a projectile may be fired vertically from the earth's surface so that it shall never return, that is, for what value of v_0 will $v = 0$ when $s = \infty$ and $s_0 = R$, we see from (8)

$$-v_0^2 = -2gR \quad \text{or} \quad v_0 = \sqrt{2gR} = \sqrt{\frac{64 \times 4000}{5280}}$$

$$= 6.96 \text{ miles approximately.}$$

Ex. 2. Assuming that the radius of the moon is 1000 miles and that the acceleration of gravity towards its center is 5.3 ft./sec.², with what velocity would a body have to be projected orthogonally away from the surface of the moon so that it would never return? Here the existence of the earth and all heavenly bodies is ignored.

Ex. 3. Assuming the earth and moon at rest and ignoring the existence of all other attracting bodies, at what point between the earth and moon would an object have no weight, because the force of attraction of the two bodies is equal in amount and directed oppositely? [Assume distance 240,000 mi.]

Ex. 4. With what least initial velocity would a body have to be projected vertically upward from a point on the earth's surface on the line of centers of the moon and the earth so that it will pass the position of equilibrium mentioned in Ex. 3 and fall upon the moon?

80. Motion subject to resisting force proportional to the square of the velocity. Taking account of the presence of the air, a moving body meets with a resisting force which may be assumed proportional to the square of its velocity, if the velocity is not very great. In the case of a *falling* body near the earth's surface starting from rest at a distance s_0 above the earth's surface, the equation of motion is

$$\frac{d^2s}{dt^2} = -g + k^2v^2. \tag{1}$$

Here k^2v^2, the resisting force due to the air, is positive because it is always directed oppositely to that of the motion of the body, and we are supposing the body to be falling.

Noting that $\frac{d^2s}{dt^2}=\frac{dv}{dt}$, the differential equation of motion is

$$\frac{dv}{dt}=-g+k^2v^2;$$

or, separating the variables,

$$\frac{dv}{k^2v^2-g}=dt, \quad \text{or} \quad \frac{kdv}{k^2v^2-g}=kdt. \tag{2}$$

Remembering that $\int\frac{du}{u^2-g}=\frac{1}{2\sqrt{g}}\log\frac{u-\sqrt{g}}{u+\sqrt{g}}$, we have, on integrating both members of (2) and multiplying by $2\sqrt{g}$,

$$\log\frac{kv-\sqrt{g}}{kv+\sqrt{g}}=2k\sqrt{g}\,t+C. \tag{3}$$

Taking the exponential[1] of each member of (3), and noting the $e^{\log A}=A$, we have

$$\frac{kv-\sqrt{g}}{kv+\sqrt{g}}=e^{2k\sqrt{g}t+C}=e^{2k\sqrt{g}t}e^{C}=ce^{2k\sqrt{g}t}, \tag{4}$$

where $e^C=c$. To determine the value of c, we note that $v=0$ when $t=0$. Hence, from (4), we have $c=-1$.

$$\therefore \frac{kv-\sqrt{g}}{kv+\sqrt{g}}=-e^{2k\sqrt{g}t}.$$

Solving this for v,

$$v=\frac{\sqrt{g}}{k}\frac{1-e^{2k\sqrt{g}t}}{1+e^{2k\sqrt{g}t}}. \tag{5}$$

---

[1] The process of raising e to the power ϕ is referred to as "taking the exponential of ϕ," and the result is sometimes written "exp ϕ." This process is employed frequently when a member of an equation is the logarithm of some expression, as in equation (3). Taking the exponential of a quantity is the process inverse to that of taking the logarithm.

This determines the velocity of the falling body at any instant. It will be noticed that v, which is negative, is always less than $\frac{\sqrt{g}}{k}$ in absolute value.

To find the distance passed over we replace v in (5) by $\frac{ds}{dt}$ and integrate, obtaining

$$s = \frac{\sqrt{g}}{k}\int \frac{1 - e^{2k\sqrt{g}t}}{1 + e^{2k\sqrt{g}t}} dt. \tag{6}$$

To evaluate the integral on the right, multiply numerator and denominator of the integrand by $e^{-k\sqrt{g}t}$, giving

$$\frac{\sqrt{g}}{k}\int \frac{e^{-k\sqrt{g}t} - e^{k\sqrt{g}t}}{e^{-k\sqrt{g}t} + e^{k\sqrt{g}t}} dt = -\frac{1}{k^2}\log(e^{-k\sqrt{g}t} + e^{k\sqrt{g}t}) + c$$

since the numerator is $-\frac{1}{k\sqrt{g}}$ times the derivative of the denominator.

$$\therefore\ s = -\frac{1}{k^2}\log(e^{k\sqrt{g}t} + e^{-k\sqrt{g}t}) + c. \tag{7}$$

Noting that $s = s_0$ when $t = 0$, we find that $c = s_0 + \frac{1}{k^2}\log 2$. Putting this in (7), and solving for $s_0 - s$, the distance traveled by the body,

$$s_0 - s = \frac{1}{k^2}\log\frac{e^{k\sqrt{g}t} + e^{-k\sqrt{g}t}}{2}. \tag{8}$$

In terms of the hyperbolic cosine this takes the simple form

$$s_0 - s = \frac{1}{k^2}\log\cosh k\sqrt{g}t. \tag{9}$$

If the body is thrown *upward*, the resisting force is directed downward. Hence the differential equation of motion is

$$\frac{dv}{dt} = -g - k^2v^2. \tag{10}$$

If $v = v_0$ when $t = 0$, show that

$$\frac{1}{\sqrt{g}} \tan^{-1}\frac{kv}{\sqrt{g}} = -kt + \frac{1}{\sqrt{g}} \tan^{-1}\frac{kv_0}{\sqrt{g}};\ \text{whence}$$

$$v = \frac{\sqrt{g}}{k} \tan\left(\tan^{-1}\frac{kv_0}{\sqrt{g}} - k\sqrt{g}t\right). \tag{11}$$

This determines the velocity of the rising body. It holds while the body is rising only, that is, until $t = \frac{1}{k\sqrt{g}} \tan^{-1}\frac{kv_0}{\sqrt{g}}$. Call it t_0. After that, the equation of motion of the falling body considered in the first place, begins to hold with the understanding that in (5) we supposed $v = 0$ when $t = 0$, while now $v = 0$ when $t = t_0$. Show that in this case (5) must be replaced by

$$v = \frac{\sqrt{g}}{k} \frac{1 - e^{2k\sqrt{g}(t - t_0)}}{1 + e^{2k\sqrt{g}(t - t_0)}}. \tag{5'}$$

To find the distance passed over, we replace v by $\frac{ds}{dt}$ in (11), and integrate, obtaining

$$s = \frac{1}{k^2}\log\cos\left(\tan^{-1}\frac{kv_0}{\sqrt{g}} - k\sqrt{g}t\right) + c. \tag{12}$$

If $s = s_0$ when $t = 0$, $c = s_0 - \frac{1}{k^2} \log\cos\tan^{-1}\frac{kv_0}{\sqrt{g}}$.

$$\therefore s - s_0 = \frac{1}{k^2} \log \frac{\cos\left(\tan^{-1}\frac{kv_0}{\sqrt{g}} - k\sqrt{g}t\right)}{\cos\tan^{-1}\frac{kv_0}{\sqrt{g}}}$$

$$= \frac{1}{k^2} \log\left(\cos k\sqrt{g}t + \frac{kv_0}{\sqrt{g}} \sin k\sqrt{g}t\right). \tag{13}$$

This determines the position of the rising body until $t = \frac{1}{k\sqrt{g}} \tan^{-1}\frac{kv_0}{\sqrt{g}}$. Show that the height reached by the body in this time is $\frac{1}{2k^2} \log\left(\frac{k^2v_0^2}{g} + 1\right)$.

81. Resisting force is proportional to the velocity. Here the same differential equation of motion holds for a falling and a rising body; namely,

$$\frac{d^2s}{dt^2}=\frac{dv}{dt}=-g-k^2v. \tag{1}$$

If $v=v_0$, $s=s_0$ when $t=0$, show that

$$v=\left(\frac{g}{k^2}+v_0\right)e^{-k^2t}-\frac{g}{k^2}, \tag{2}$$

$$s-s_0=\frac{1}{k^2}\left(\frac{g}{k^2}+v_0\right)(1-e^{-k^2t})-\frac{k^2}{gt}. \tag{3}$$

In this case also, the velocity of a falling body is restricted to be less than a certain amount in absolute value. What is this limiting value? Does it depend upon v_0?

If the resisting force is proportional to the velocity and there is *no other force acting*, the differential equation of motion is

$$\frac{d^2s}{dt^2}=\frac{dv}{dt}=-k^2v. \tag{4}$$

If $v=v_0$, $s=0$ when $t=0$, show that

$$v=v_0e^{-k^2t}, \tag{5}$$

$$s=\frac{v_0}{k^2}(1-e^{-k^2t}). \tag{6}$$

The resistance offered by water to a body moving through it at a moderate velocity is approximately proportional to the velocity.

Ex. 1. A man in a boat stops rowing when moving 6 miles/hour. One minute later the velocity of the boat is three miles/hour. What is the velocity 2 minutes after the rowing ceased? 5 minutes after? How far did the boat go during 1, 2, 5 minutes respectively after the rowing ceased?

Since 6 miles/hour = 528 ft./min. and 3 miles/hour = 264 ft./min., we have, on putting $v_0 = 528$, $v = 264$, $t = 1$ in (5),

$$e^{-k^2} = \tfrac{1}{2}. \quad \therefore e^{k^2} = 2 \quad \text{and} \quad k^2 = \log 2 = 0.693.$$

For $t = 2$, $v = 528(\frac{1}{2})^2 = 132$ ft./min.

To find s, use is made of (6).

For $t = 1$, $s = \dfrac{528}{0.693}(1 - \frac{1}{2}) = 381$ ft. approximately.

Ex. 2. If the resisting force is proportional to the cube of the velocity, and there is no other force acting, show that if a body is given an initial velocity v_0, the distance traveled by it in the time t is $s = \dfrac{\sqrt{2\,k^2 v_0^2 t + 1} - 1}{k^2 v_0}$.

82. Flexible homogeneous wire. Let AOB be a flexible homogeneous wire fastened at A and B, and acted upon by gravity only. What is the equation of the curve assumed by it, when in equilibrium?

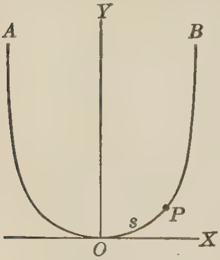

FIG. 77

We shall take O, the lowest point on the curve, as origin, and the tangent at O, as the axis of x.

Let P be a second point on the curve at a distance s from O, measured along the curve. Let its coördinates be (x, y).

When the wire is in equilibrium, there is a tension at each point directed along the tangent to the curve at that point. Imagine the portion OP cut free from the rest of the wire. To keep it in equilibrium, we must supply at O and P the tangential forces H and T respectively, equal to the tensions that existed at those points before cutting and directed as in Fig. 78. The re-

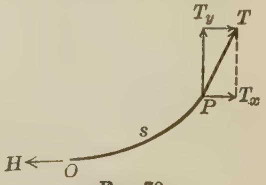

FIG. 78

leased portion OP is now in equilibrium, subject to the forces H, T, and the weight of the portion. If w is the weight of unit length of the wire, the weight of the portion is ws.

Let the horizontal and vertical components of T be T_x and T_y respectively. Then, since there is equilibrium,

$$T_x = -H, \quad T_y = -ws. \quad \therefore \frac{T_y}{T_x} = \frac{w}{H}s.$$

Noting that w and H are constants, we shall write $\frac{w}{H} = \frac{1}{a}$. Moreover, $\frac{T_y}{T_x} = \frac{dy}{dx}$, the slope of the curve at P. Hence the differential of the equation of the curve is

$$\frac{dy}{dx} = \frac{s}{a}. \tag{1}$$

This involves three variables, x, y, and s. We can get rid of s by differentiating both members of (1) and remembering that $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Doing this, we have

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2}$$

In this differential equation, the dependent variable does not appear, its derivatives only appearing. Letting $\frac{dy}{dx} = p$, then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, and

$$\frac{dp}{dx} = \frac{1}{a}\sqrt{1 + p^2}. \tag{3}$$

Separating the variables, and integrating,

$$\int \frac{dp}{\sqrt{1+p^2}} = \frac{1}{a}\int dx, \text{ or}$$

$$\log\left(p + \sqrt{1+p^2}\right) = \frac{x}{a} + c. \tag{4}$$

Since $p=0$ when $x=0$, $c=0$ in (4). Taking exponentials,

$$p+\sqrt{1+p^2}=e^{\frac{x}{a}}, \quad \text{or} \quad \sqrt{1+p^2}=e^{\frac{x}{a}}-p.$$

Squaring both sides and solving for p,

$$p=\tfrac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}}). \tag{5}$$

Remembering that $p=\dfrac{dy}{dx}$, we have on integrating (5),

$$y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})+c.$$

Since $y=0$ when $x=0$, $c=-a$. Hence equation of the curve is

$$y+a=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})=a\cosh\frac{x}{a}, \tag{6}$$

which is the equation of a catenary, when the origin is at the vertex. A flexible chain also assumes this shape, approximately, when suspended from two points. The name "catenary" is derived from the Latin word "catena," a chain.

83. Parabolic cable. In the case considered in the previous section the weight of any portion, OP, of the wire was proportional to s, the length of OP. Suppose now there is suspended from the wire or cable a weight, for example a bridge CD, and in such a way that the total weight carried by any portion OP is proportional to OM, the horizontal projection of OP. What is the equation of the curve assumed by the wire or cable in this case?

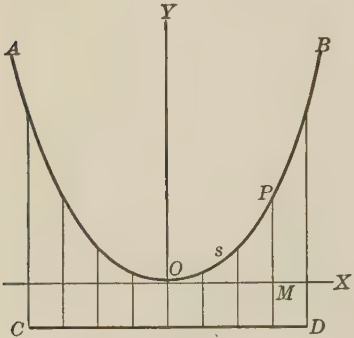

FIG. 79

Proceeding as in the previous case, the detached portion OP will be in equilibrium under the forces H at O, T at P, and the weight carried by OP. If the weight carried per unit of projection of OP on OX is W, the weight carried by OP is Wx, the coördinates of P being (x, y).

If, as before, T_x and T_y are the horizontal and vertical projections of T,

$$T_x = -H,\ T_y = -Wx.$$

$$\therefore \frac{T_y}{T_x} = \frac{W}{H}x = kx,$$

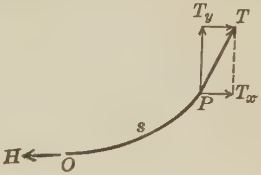

FIG. 80

where $\frac{W}{H} = k$ is constant. Hence the differential equation of the curve assumed by the wire or cable is

$$\frac{dy}{dx} = kx. \tag{1}$$

Multiplying both sides by 2 and integrating,

$$2y = kx^2 + c.$$

Since $y = 0$ when $x = 0$, $c = 0$. Hence the equation of the curve is

$$2y = kx^2, \tag{2}$$

which is the equation of a parabola with the vertex at the origin.

84. Chemical applications. It is known that sugar, in the presence of certain chemicals, is converted into other substances. If x is the quantity of sugar that has been converted during any time t, then $\frac{dx}{dt}$ is the rate of conversion at the instant t. It is found that this rate is proportional to the quantity of unconverted sugar at that instant. If a is the quantity of sugar started with, the differential equation to determine x as a function of t is

$$\frac{dx}{dt} = k(a - x),$$

where k is the factor of proportionality.

To solve, separate the variables

$$\frac{dx}{a-x} = k\,dt.$$

Integrating, $$\log(a-x) = -kt + C = -kt + e^c.$$

Taking exponentials of both sides,

$$a - x = ce^{-kt}.$$

Since $x = 0$ when $t = 0$, $c = a$, and we have finally

$$x = a(1 - e^{-kt}).$$

Ex. In some chemical reactions, the rate of conversion of a substance at any instant is found to be proportional to the square of the quantity of the substance untransformed at that instant. Show that the amount of the substance that is transformed in the time t is $x = \dfrac{a^2 kt}{akt+1}$, where k is the factor of proportionality and a is the amount of the substance started with.

85. Growth function. Experiments with colonies of animals, such as fruit flies, and other considerations have led to an hypothesis, giving at least a first approximation for the rate of normal growth of the population of a stable aggregate of beings, such as in the case of a nation over a period of years. Since the number of persons that could possibly people such a country is limited, let the largest possible number of inhabitants be N. If at any time t the number of inhabitants is x, the hypothesis is that the rate of growth of the population is proportional to the product of x and $N - x$, that is,

$$\frac{dx}{dt} = kx(N-x), \tag{1}$$

where k is the factor of proportionality. Separating the variables in (1),

$$\frac{dx}{x(N-x)} = k\,dt.$$

Breaking up the left-hand member into a sum of partial fractions, we have, after multiplying both members by N and integrating,

$$\log \frac{x}{N-x} = Nkt + e^{c} \quad \text{or} \quad \frac{x}{N-x} = ce^{Nkt}.$$

Or, solving for x,

$$x = \frac{cN}{c + e^{-Nkt}}. \tag{2}$$

The fraction x is known as the *growth function.* The values of the constants c, N, k, can be determined from census data at three periods of time.

It is interesting to note that the growth function (2) conforms very well as a first approximation to facts observed in the case of the population of colonies of animals or of nations.

If, instead of using the constant factor k in (1), a variable factor $f(t)$ is introduced, a more general type of growth function is obtained, which for proper choices of $f(t)$ is found to fit observed facts even better. It will be left as an exercise to show that in this case the growth function has the form

$$x = \frac{cN}{c + e^{-N\int f(t)\,dt}}.$$

For further information on this subject see the following article by Raymond Pearl and Lowell J. Reed:

On the rate of growth of the population of the United States since 1790 and its mathematical representation, *Proceedings of the National Academy of Sciences*, pp. 275–288, Vol. 6, 1920.

86. Electrical application. We shall conclude the list of illustrative exercises leading to the solution of simple differential equations with one on the subject of electricity. In an electric circuit containing a cell producing a constant electromotive force E and a coil of constant self-inductance L, the total resistance is r, also a constant. As soon as the circuit is closed, the current begins to rise.

If i is the value of the current at any time t ($i=0$ when $t=0$), the induced electromotive force is $L\frac{di}{dt}$. This opposes E. The net electromotive force is thus $E-L\frac{di}{dt}$.

$$\therefore E-L\frac{di}{dt}=ri, \text{ or}$$

$$L\frac{di}{dt}+ri=E, \tag{1}$$

which is the differential equation determining i as a function of t. Separating the variables

$$\frac{Ldi}{ri-E}=-dt, \text{ or } \frac{rdi}{ri-E}=-\frac{r}{L}dt.$$

Integrating, we have

$$\log(ri-E)=-\frac{rt}{L}+e^c.$$

Taking exponentials,

$$ri-E=ce^{-\frac{rt}{L}}.$$

Since $i=0$ when $t=0$, $c=-E$.

$$\therefore i=\frac{E}{r}(1-e^{-\frac{rt}{L}}). \tag{2}$$

The current approaches $\frac{E}{r}$ as a limit.

CHAPTER X

DEFINITE INTEGRAL

87. Limit of a sum. Heretofore we have considered integration as the process inverse to that of differentiation. Historically the integral calculus had its initial development from a different point of view. We shall endeavor to present in an elementary manner, without insisting upon the proofs of important steps, the notion of a definite integral as the limit of a sum.

We shall start out with the geometrical problem: Given the parabola $y = x^2 + 1$, what is the area under the arc of this curve from $x = 0$ to $x = 3$? In Fig. 81, this area is that included between the arc AC, the axis of x, and the ordinates OA and BC.

As a first approximation, we can take the area of the trapezoid $OACB$. This is $\dfrac{3(1+10)}{2} = 16.5$, which is evidently too large.

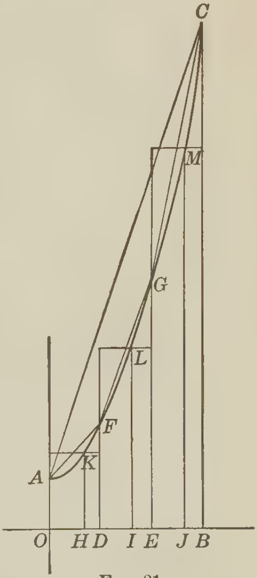

FIG. 81

Now divide OB into three equal parts, each one unit in length, and at the points of subdivision D and E erect ordinates to the curve. The upper extremities of these are $F = (1, 2)$ and $G = (2, 5)$, respectively. Draw the chords AF, FG, GC. The area under the broken line made up of these chords may be taken as a second approximation of the desired area. It is made up of three trapezoids, the sum of whose areas is 12.5.

If we halve each of the unit segments of line OB, and at all the points of subdivision erect ordinates to the curve and join the adjacent upper extremities of these with straight lines, the area under this new broken line is easily found to be 12.125.

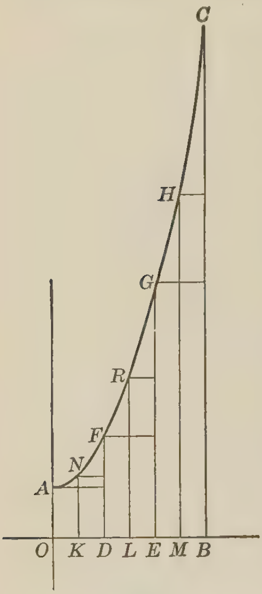

FIG. 82

If we halve each of these smaller segments of OB and proceed as before, the area under the new broken line is readily calculated to be 12.03125.

If we divide each of the unit segments of OB into ten equal parts, the area under the corresponding broken line is found to be 12.0070.

The successive numbers

$$16.5, \quad 12.5, \quad 12.125, \quad 12.03125, \quad 12.0070$$

suggest the number 12 as a limit. We know for a fact that the area under AC is equal to 12 (§ 38).

As a second method of approximation, consider the sum of the rectangles DA, EF, BG in Fig. 82, each of which has for base the corresponding segment of OB, and for altitude the ordinate at the left-hand end of the segment.

This sum is $1 + 2 + 5 = 8$.

As before, halve each of the three segments OD, DE, and EB, and consider the sum of the resulting rectangles obtained by the method just described, viz.,

$$KA, \ DN, \ LF, \ ER, \ MG, \ BH.$$

This turns out to be 9.875.

If we halve each of these smaller segments of OB and consider the sum of the rectangles obtained in the prescribed manner, this turns out to be 11.21876.

If we divide each of the unit segments of OB into ten equal parts and consider the sum of the rectangles obtained in the prescribed manner, this turns out to be 11.555.

If each of the unit segments of OB is divided into a hundred equal parts, the sum of the corresponding rectangles is 11.955050.

By taking a larger and still larger number of subdivisions of OB, the successive numbers obtained for the sums of the rectangles

$$8, \quad 9.875, \quad 11.21876, \quad 11.555, \quad 11.955050, \cdots$$

are found to approximate more and more closely to 12. Inspection of Figs. 81 and 82 will show why the sum of the trapezoids gives a better approximation than the sum of the rectangles, for the same number of subdivisions of OB. But the point to be emphasized is that while we have only indicated it, it can be proved that, in each case, the approximating sum can be made as near 12 as may be desired, by taking the number of subdivisions of OB large enough.

Since the area of a trapezoid is equal to the product of its altitude by half the sum of its bases, each of the trapezoids in Fig. 81 may be replaced by a rectangle having for altitude half the sum of the parallel sides of the trapezoid and for base the segment of OB on which the trapezoid stands. Thus, in Fig. 81,

the trapezoid $DFAO$ may be replaced by the rectangle having $HK = \frac{1}{2}(OA + DF)$ for altitude and OD for base;

the trapezoid $EGFD$ may be replaced by the rectangle having $IL = \frac{1}{2}(DF + EG)$ for altitude and DE for base;

and so on.

Here the ordinate HK corresponds to the abscissa $\sqrt{0.5}$, which is >0.5 but lies between 0 and 1; the ordinate IL corresponds to the abscissa $\sqrt{2.5}$ which is >1.5 but lies between 1 and 2; the ordinate JM corresponds to the abscissa $\sqrt{6.5}$ which is >2.5 but lies between 2 and 3.

A third type of approximation for the area under the arc of the curve is to consider the rectangles each having a segment of OB for base and for altitude the ordinate of the curve corresponding to the point at the right-hand extremity of the segment. Fig. 83 illustrates it. Comparing this with Fig. 82, it will be seen that at each stage the rectangles in the two figures are precisely the same, except that the one to the extreme left in Fig. 82 is missing in Fig. 83, while a compensating rectangle at the extreme right appears in Fig. 83 which is missing in Fig. 82. Corresponding to the successive stages of approximation considered in the second case, the values of the sum of the areas of the rectangles in the third case are readily found to be

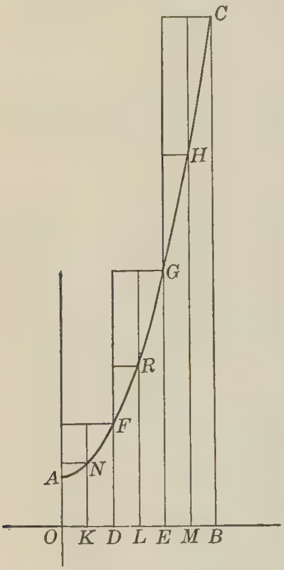

Fig. 83

$$17, \quad 14.375, \quad 13.46875, \quad 12.455, \quad 12.045050.$$

Here again the sum of the rectangles is seen to approximate 12 as the number of rectangles is increased.

This simple exercise carried through with considerable detail is intended to illustrate the following fact:

Let $f(x)$ be a continuous function in the interval (a, b); *i.e.* $a \leq x \leq b$. In Fig. 84, CD represents the part of the curve $y = f(x)$ corresponding to the values of x in the interval (a, b).

Let A be the point on the axis of x whose abscissa is a, and B the point on this axis whose abscissa is b. Let AB be divided into n equal parts, the points of subdivision being X_1, $X_2, \cdots, X_{n-1}$. At each of these points, and at A and B, which take the place of X_0 and X_n respectively, draw the corresponding ordinate to the curve, obtaining

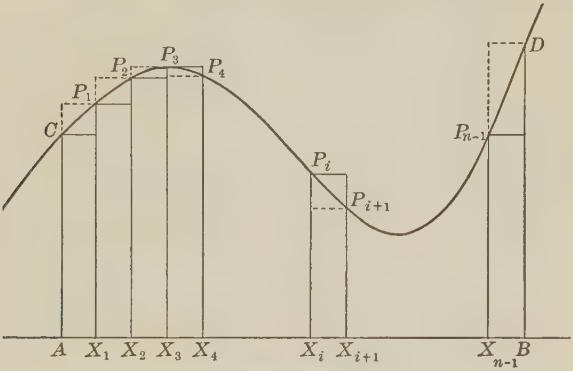

FIG. 84

$$AC, \quad X_1P_1, \quad X_2P_2, \cdots, X_{n-1}P_{n-1}, \quad BD.$$

Changing the order adopted in the illustrative exercise, we shall consider

(a) the rectangles, each having a segment of AB for base, and the ordinate corresponding to the left-hand extremity of the segment for altitude; in Fig. 84 these are the rectangles

$$X_1C,\ X_2P_1, \cdots, X_{i+1}P_i, \cdots, BP_{n-1};$$

(b) the rectangles, each having a segment of AB for base and the ordinate corresponding to the right-hand extremity of the segment for altitude; in Fig. 84 these are the rectangles

$$AP_1,\ X_1P_2, \cdots, X_iP_{i+1}, \cdots, X_{n-1}D;$$

(c) the trapezoids, each having a segment of AB for base and the ordinates at the extremities of the segment for parallel sides; as noted in the illustrative exercise, such a trapezoid is equal in area to a rectangle which has the segment of AB for base and the ordinate corresponding to some point along the segment for altitude.

The fact we wish to emphasize is that it can be proved that, *by making n sufficiently large, we can make the sum of the areas of each set of rectangles (a), (b), and (c) differ from the area under CD by as small a quantity as desired.* Or using the language of limits, it can be proved that *as n approaches ∞ the limit of the sum of the areas of each set of rectangles (a), (b), and (c) is the area under CD.*

88. Fundamental theorem of the integral calculus. Definite integral. The geometrical theorem stated at the close of the previous section suggests a most important algebraic one.

Let the abscissas of $X_1, X_2, \cdots, X_i, \cdots, X_{n-1}$ in Fig. 84 be $x_1, x_2, \cdots, x_i, \cdots, x_{n-1}$ respectively, and besides let $a = x_0$ and $b = x_n$. Then the sum of the areas of the rectangles (a) of § 87 is equal to

$$f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_i)(x_{i+1} - x_i) + \cdots \\ + f(x_{n-1})(x_n - x_{n-1}).$$

It is convenient to represent the sum of a set of terms by the following symbol:

$$\sum_{i=0}^{5} a_i = a_0 + a_1 + a_2 + a_3 + a_4 + a_5,$$

$$\sum_{k=3}^{7} u_k = u_3 + u_4 + u_5 + u_6 + u_7,$$

$$\sum_{l=0}^{n-1} f(x_l) = f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}).$$

Here $\sum$ is the capital Greek sigma, corresponding to our S. The sum of the areas of the rectangles (a) of § 87 can then be expressed in the compact form

$$\sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i). \tag{1}$$

Similarly the sum of the areas of the rectangles (b) of § 87 can be expressed as

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \quad \text{or} \quad \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i); \tag{2}$$

and the sum of the areas of the trapezoids or rectangles (c) of § 87 can be expressed as

$$\sum_{i=0}^{n-1} f(\bar{x}_i)(x_{i+1} - x_i), \tag{3}$$

where $\bar{x}_i$ is a value of x intermediate between x_i and x_{i+1}, determined by the relation $2f(\bar{x}_i) = f(x_i) + f(x_{i+1})$.

The theorem at the close of § 87 states that each of the sums (1), (2), (3) has the same limit as n approaches ∞, and this common limit is the numerical value of the area under the curve CD, Fig. 84.

While we have introduced the subject of the existence of the limit for the sums (1), (2), (3), and the evaluation of the limit in a geometrical way, they are fundamentally arithmetic or algebraic. We shall therefore give a purely algebraic statement of the

FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS: *Let $f(x)$ be a continuous function of x throughout the interval (a, b). Divide this interval into n equal subintervals, the extremities of* *which are the various pairs of adjacent numbers in the set* $x_0 = a, x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_{n-1}, x_n = b$, *and form the sum*

$$\sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_{n-1})(x_n - x_{n-1}). \qquad (4)$$

As n increases without limit this sum approaches a definite limit.

It is customary to designate this limit by $\int_a^b f(x)\,dx$, where the Greek sigma $\sum$ is replaced by the Latin long *s* "$\int$," which style of type was in use at the time when this symbol was introduced, and the increment $x_{i+1} - x_i$, which may be written Δx_i, is replaced by "dx"; a and b, the limits of the interval over which x is taken, are also brought into evidence in the symbol. So the fundamental theorem may be expressed symbolically

$$\underset{n\to\infty}{L} \sum_{i=0}^{n-1} f(x_i)\Delta x_i = \int_a^b f(x)\,dx. \qquad (5)$$

This is known as *the definite integral of $f(x)$ taken between the limits a and b*; a is said to be the *lower* limit and b the *upper* limit of the integral. The definite integral defined here as the limit of a sum can be shown to be identical with the definite integral defined in § 38.

The statement of the fundamental theorem given above is not as general as it can be made. Generalizations in two respects should be noted here.

1. It is not necessary to have all the subintervals of the interval (a, b) of equal size. It can be shown that the limit of (4) is the same for *any* law of division of the interval (a, b) by which each of the subintervals ultimately becomes as small as one pleases, that is, by which it is possible to name an n such that each of the corresponding subintervals will be less than any previously named amount, no matter how small.

2. In $f(x_i)$, x_i may be replaced by any value of x in the ith interval (x_i, x_{i+1}) and the value of the limit is left unaltered. Thus the limits of (2) and (3) are the same as that of (1).

This second generalization suggests a very important theorem on the limit of a sum, to which attention will be called here. Δx_i is an infinitesimal in the definite integral, since it approaches zero as n increases indefinitely. Hence $f(x_i)\Delta x_i$ also is an infinitesimal in (5). The definite integral (5) is thus the limit of the sum of infinitesimals whose number grows indefinitely as each of the infinitesimals approaches zero. The fact we wish to emphasize is that *if*

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

is the sum of a set of positive infinitesimals which approaches a limit as n increases indefinitely, and if

$$\sum_{i=1}^{n} \beta_i = \beta_1 + \beta_2 + \cdots + \beta_n$$

is the sum of a second set of infinitesimals, such that

$$\underset{n \to \infty}{L} \frac{\beta_i}{\alpha_i} = 1, \textit{ for all values of } i, \tag{6}$$

then

$$\underset{n \to \infty}{L} \sum_{i=1}^{n} \beta_i = \underset{n \to \infty}{L} \sum_{i=1}^{n} \alpha_i.$$

This is known as *Duhamel's theorem.* Frequent use will be made of it.

89. Value of definite integral. As defined in § 88, the definite integral is the limit of a sum. While, as illustrated in § 87, it may be possible to approximate this fairly closely in the case of a given function $f(x)$, by taking n sufficiently large, we can frequently evaluate a definite integral by making use of the fact that $\int_a^b f(x)\,dx$ is numerically equal to the area under the curve $y=f(x)$ between the ordinates $x=a$ and $x=b$. In § 38, we saw that if $F(x)$ is any indefinite integral or antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Hence this also must be the value of the definite integral defined as the limit of a sum.

90. Properties of definite integral. The definition of a definite integral as the limit of a sum leads to many important facts and applications.

The following properties follow readily from the definition of a definite integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{1}$$

If $b > a$, then $\Delta x_i \equiv x_{i+1} - x_i$ is positive, for all values of i, in the sum of which the integral on the left is the limit, and Δx_i is negative in the sum of which the integral on the right is the limit. Hence (1) follows.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx. \tag{2}$$

If b lies between a and c, the interval (a, c) is equal to the sum of the intervals (a, b) and (b, c), and (2) follows from the definition of a definite integral.

If c lies between a and b, the interval $(a, b) = (a, c) + (c, b)$.

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx.$$

Hence (2) follows in this case.

If a lies between b and c, the interval $(b, c) = (b, a) + (a, c)$.

$$\therefore \int_b^c f(x)\,dx = \int_b^a f(x)\,dx + \int_a^c f(x)\,dx$$

$$= -\int_a^b f(x)\,dx + \int_a^c f(x)\,dx.$$

Adding $\int_a^b f(x)\,dx$ to each member of this relation, (2) follows.

Making use of (1), formula (2) can be put in the more symmetrical form,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = 0. \tag{2'}$$

REMARK 1. — The value of the definite integral $\int_a^b f(x)\,dx$ evidently depends upon the form of the function $f(x)$ and the values of a and b, but is independent of the choice of the letter used to represent the variable. Thus

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du.$$

REMARK 2. — If $b > a$ and $f(x)$ has the same algebraic sign for all values of x in the interval, $\int_a^b f(x)\,dx$ has the same algebraic sign as $f(x)$. Geometrically this means that when $f(x)$ is positive in an interval, the corresponding area under the curve is given as positive by the definite integral; while when $f(x)$ is negative in the interval, the corresponding area under the curve (which is actually above the curve, but below the axis of x) is given as negative. Thus

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big]_0^{\frac{\pi}{2}} = 0 - (-1) = 1,$$

$$\int_{-\frac{\pi}{2}}^0 \sin x\,dx = -\cos x\Big]_{-\frac{\pi}{2}}^0 = -1 - 0 = -1.$$

Hence we are not surprised to have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 - 0 = 0.$$

This suggests the important fact that whenever the curve $y = f(x)$ is symmetrical with respect to the origin, that is, whenever $f(-x) = -f(x)$ [in which case $f(x)$ is said to be an *odd function*],

$$\int_{-a}^{a} f(x)dx = 0, \text{ for } a \text{ any number.}$$

For letting $x = -u$, whence $dx = -du$

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_{a}^{0} f(u)du = -\int_{0}^{a} f(u)du$$
$$= -\int_{0}^{a} f(x)dx.$$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

Ex. — In an analogous manner prove that if $f(-x) = f(x)$ [in which case $f(x)$ is said to be an *even function*],

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

91. Theorem of the mean. As all possible values in the interval (a, b) are assigned to x, $f(x)$ assumes a set of values among which there is a smallest one, m, and a largest one, M. Evidently the value of the integral $\int_a^b f(x)dx$, which is equal to the area under the curve $y = f(x)$ from $x = a$ to $x = b$, lies between the values

$$(b - a)m \quad \text{and} \quad (b - a)M.$$

But this area under the curve has a definite value A. If $\frac{A}{(b-a)} = k$, the value of the integral is $(b - a)k$, where $m < k < M$. Since $f(x)$ is continuous, it can be proved that corresponding to k, intermediate between m and M, there is at least one value of x, say $\bar{x}$, such that $f(\bar{x}) = k$. So we may write

$$\int_a^b f(x)dx = (b - a)f(\bar{x}),$$

where $\bar{x}$ is some number intermediate between a and b. This formula is known as the *theorem of the mean for integration.*

It may not be practicable to determine the value of $\bar{x}$. In the particular case illustrated in Fig. 85, $\bar{x}$ may have any one of three values. But this theorem is of great importance for theoretical purposes.

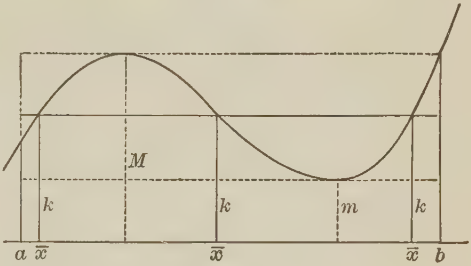

FIG. 85

If θ is a positive number less than unity; *i.e.* $0<\theta<1$, we can write $\bar{x}=a+\theta(b-a)$, and the theorem takes the form

$$\int_a^b f(x)dx=(b-a)f[a+\theta(b-a)].$$

If now we write $b-a=h$, whence $b=a+h$, we have the convenient form

$$\int_a^{a+h} f(x)dx=hf(a+\theta h).$$

92. Improper integrals. In the definition of a definite integral as the limit of the sum, it was assumed that $f(x)$ is continuous throughout the interval (a, b) and that both a and b are finite. If either of these assumptions does not hold, further definition of what shall be understood by the definite integral is necessary. In either case, the integral is said to be an *improper* one.

A. If $\underset{h\to 0}{L} f(a+h)=\infty$,

$$\int_a^b f(x)dx \text{ is defined as equal to } \underset{h\to 0}{L}\int_{a+h}^b f(x)dx.$$

If $\underset{h \to 0}{L} f(b-h) = \infty$,

$\int_a^b f(x)dx$ is defined as equal to $\underset{h \to 0}{L} \int_a^{b-h} f(x)dx$.

If $\underset{h \to 0}{L} f(c \pm h) = \infty$, where $a < c < b$,

$\int_a^b f(x)dx$ is defined as equal to

$$\underset{h \to 0}{L} \int_a^{c-h} f(x)dx + \underset{h \to 0}{L} \int_{c+h}^{b} f(x)dx.$$

The following exercises will illustrate.

Ex. 1 $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. Here $\underset{h \to 0}{L} \frac{1}{\sqrt{1-(1-h)^2}} = \infty$.

$$\int_0^{1-h} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big]_0^{1-h} = \sin^{-1}(1-h) - \sin^{-1}0$$
$$= \sin^{-1}(1-h).$$

$$\underset{h \to 0}{L} \sin^{-1}(1-h) = \sin^{-1}1 = \frac{\pi}{2} \quad \therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

Ex. 2. $\int_0^1 \frac{dx}{x}$. Here $\underset{h \to 0}{L} \frac{1}{h} = \infty$.

$$\int_h^1 \frac{dx}{x} = \log x\Big]_h^1 = \log 1 - \log h = -\log h.$$

Since $\log h$ does not approach a definite finite value as h approaches zero, we cannot evaluate $\int_0^1 \frac{dx}{x}$.

If we admit infinite values and note that then $\underset{h \to 0}{L} \log h = -\infty$, we may say that $\int_0^1 \frac{dx}{x} = \infty$.

Ex. 3. $\int_{-1}^1 \frac{dx}{x^2}$. If we evaluate this blindly, we find $\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big]_{-1}^1 = -1-1 = -2$, a result obviously in conflict with Remark 2 of § 90; for $\frac{1}{x^2}$ is always positive. The reason is that $\frac{1}{x^2}$ approaches ∞ as x approaches zero. So we must examine

$$\underset{h \to 0}{L} \int_{-1}^{-h} \frac{dx}{x^2} \text{ and } \underset{h \to 0}{L} \int_{h}^{1} \frac{dx}{x^2}.$$

Each of these is infinite.

B. If either of the limits of the definite integral is infinite, we shall define

$$\int_a^\infty f(x)dx \text{ as equal to } \underset{b \to \infty}{L} \int_a^b f(x)dx,$$

$$\int_{-\infty}^b f(x)dx \text{ as equal to } \underset{a \to -\infty}{L} \int_a^b f(x)dx,$$

$$\int_{-\infty}^\infty f(x)dx \text{ as equal to } \underset{a \to -\infty}{L} \int_a^c f(x)dx + \underset{b \to \infty}{L} \int_c^b f(x)dx,$$

where c is a convenient finite number.

The following exercises will illustrate.

Ex. 1. $\int_0^\infty \frac{dx}{x^2+k^2}$. $\int_0^b \frac{dx}{x^2+k^2} = \frac{1}{k} \tan^{-1} \frac{x}{k}\Big]_0^b = \frac{1}{k} \tan^{-1} \frac{b}{k}$.

Since $\underset{b \to \infty}{L} \tan^{-1} \frac{b}{k} = \frac{\pi}{2}$, $\int_0^\infty \frac{dx}{x^2+k^2} = \frac{\pi}{2k}$.

Ex. 2. $\int_1^\infty \frac{dx}{x}$. $\int_1^b \frac{dx}{x} = \log x\Big]_1^b = \log b$.

Since $\underset{b \to \infty}{L} \log b = \infty$, $\int_1^\infty \frac{dx}{x} = \infty$.

Ex. 3. $\int_{-\infty}^0 \sin x\, dx$. $\int_a^0 \sin x\, dx = -\cos x\Big]_a^0 = \cos a - 1$.

Since $\underset{a \to -\infty}{L} \cos a$ does not exist, $\int_{-\infty}^0 \sin x\, dx$ cannot be evaluated.

EXERCISES

Evaluate the following improper definite integrals.

1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

2. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$.

3. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x^4}}$.

4. $\displaystyle\int_0^a \frac{dx}{\sqrt{a-x}}$.

5. $\displaystyle\int_0^a \frac{dx}{\sqrt{(a-x)^3}}$.

6. $\displaystyle\int_{-a}^a \frac{dx}{a+x}$.

7. $\displaystyle\int_{-a}^a \frac{dx}{a^2-x^2}$.

8. $\displaystyle\int_{-a}^a \frac{dx}{(a^2-x^2)^2}$.

9. $\displaystyle\int_0^4 \frac{dx}{(x-2)^{\frac{1}{3}}}$.

10. $\displaystyle\int_0^4 \frac{x\,dx}{\sqrt{4-x^2}}$.

11. $\displaystyle\int_0^4 \frac{dx}{4-x^2}$.

12. $\displaystyle\int_4^\infty \frac{dx}{4-x^2}$.

13. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$.

14. $\displaystyle\int_0^\infty e^{-x}\,dx$.

15. $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

16. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2}$.

17. $\displaystyle\int_0^a \frac{dx}{(a-x)^n}$. Consider separately the cases $n<1$, $n=1$, $n>1$.

18. $\displaystyle\int_{2a}^\infty \frac{dx}{(x-a)^n}$. Consider separately the cases $n<1$, $n=1$, $n>1$.

CHAPTER XI

APPLICATIONS

93. Area; rectangular coördinates. As noted before, §§ 38 and 89, the area under the curve $y = f(x)$, from $x = a$ to $x = b$, is

$$A = \int_a^b f(x)\,dx. \tag{1}$$

Similarly the area bounded by the curve $x = \phi(y)$, the axis of y, and the lines $y = a$ and $y = b$ is

$$A = \int_a^b \phi(y)\,dy. \tag{1'}$$

Ex. 1. Find the total area of the curve $x^2 = y^3 - y^4$.

It will be left as an exercise for the student to show that the curve is as represented in Fig. 86.

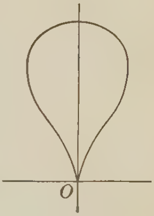

FIG. 86

Owing to symmetry, the area of the curve is evidently

$$A = 2\int_0^1 y^{\frac{3}{2}}(1 - y)^{\frac{1}{2}}\,dy.$$

To integrate this let $y = \sin^2\theta$; then $dy = 2\sin\theta\cos\theta\,d\theta$, and

$$A = 4\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta\,d\theta = \frac{\pi}{8}.$$

Ex. 2. Find the area common to the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 5$.

The curves intersect in the points $P = (1, 2)$ and $Q = (1, -2)$, Fig. 87. Owing to symmetry, the required area is twice that of OPR.

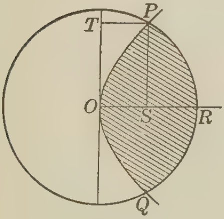

FIG. 87

Using formula (1), we find the area of *OPR* by making use of the fact that it is equal to the sum of the areas of *OPS* and *SPR*.

$$\text{Area } OPS = \int_0^1 \sqrt{4x}\,dx = 2\int_0^1 \sqrt{x}\,dx = \tfrac{4}{3}x^{\frac{3}{2}}\Big]_0^1 = \tfrac{4}{3}.$$

$$\begin{aligned}\text{Area } SPR &= \int_1^{\sqrt{5}} \sqrt{5 - x^2}\,dx \\ &= \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\Big]_1^{\sqrt{5}} \\ &= -1 + \frac{5}{2}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{5}\right).\end{aligned}$$

$$\therefore \text{ Area } OPR = \frac{1}{3} + \frac{5}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{5}}\right).$$

$$\text{Required area} = \frac{2}{3} + 5\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{5}}\right).$$

We can also solve the problem by using formula (1′). In this case we must find twice the area of *TPRO* and subtract from it twice the area of *TPO*.

$$\therefore \text{ Required area} = 2\int_0^2 \left(\sqrt{5 - y^2} - \frac{y^2}{4}\right)dy = \frac{2}{3} + 5\sin^{-1}\frac{2}{\sqrt{5}}.$$

The two answers are equal. In order to substantiate this statement, it must be shown that

$$\sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{5}}.$$

This will be left as an exercise.

Ex. 3. Find the area of the loop of the folium of Descartes (§ 23) whose equations are

$$x = \frac{3at}{1 + t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

Here $dx = \dfrac{3a(1 - 2t^3)}{(1 + t^3)^2}dt$. The loop extends from $t = 0$ to $t = \infty$. Hence, using formula (1),

$$\text{Area of loop} = 9a^2\int_0^\infty \frac{(1-2t^3)t^2}{(1+t^3)^3}dt.$$

To evaluate this let $1 + t^3 = u$; then $t^3 = u - 1$, and $3t^2dt = du$.

$$\therefore \text{ Area} = 3a^2\int_1^\infty \frac{3-2u}{u^3}du = -\frac{9a^2}{2u^2} + \frac{6a^2}{u}\bigg]_1^\infty = -\frac{3a^2}{2}.$$

The reason for the negative sign of the result is that as t grows from 0 to ∞ the loop is traced in the direction of the arrows, Fig. 88. Formula (1) gives us the area under OAB minus the area under OCB.

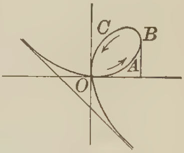

FIG. 88

If formula (1′) is used, the area turns out to be positive, as an inspection of Fig. 88 suggests.

The student should find the area, using formula (1′).

EXERCISES

1. Find the entire area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Also use the parametric equations $x = a \cos \theta$, $y = a \sin \theta$.

2. Find the area between the hyperbola $x^2 - y^2 = a^2$ and the line $x = x_1$, where $x_1 > a$.

3. Find the entire area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. Also use the parametric equations $x = a \cos^3 t$, $y = a \sin^3 t$.

4. Find the entire area of the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

5. Find the area between the cissoid $y^2 = \frac{x^3}{2a - x}$ and its asymptote $x = 2a$.

6. Find the area under one arch of $y = \sin x$ from $x = 0$ to $x = \pi$.

7. Find the area under the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $x = -x_1$ to $x = x_1$.

8. Find the area between the witch $y = \frac{8a^3}{x^2 + 4a^2}$ and its asymptote.

9. Find the area under the witch from $x = 0$ to $x = 2a$.

10. Find the area between the witch and the parabola $x^2 = 4ay$.

11. Prove that the area under the parabola $y^2 = 4ax$ from the origin to the point (x_1, y_1) on the curve is equal to $\frac{2}{3}x_1y_1$.

12. Find the area in the first quadrant between the circle $x^2 + y^2 - 2ax = 0$ and the parabola $y^2 = ax$.

13. Find the area between $y = \sin x$, $y = \cos x$, and the axis of y.

14. Find the area under one arch of $x = at$, $y = a(1 - \cos t)$.

15. Find the area of one loop of $a^2y^2 = a^2x^4 - x^6$.

94. Area; polar coördinates. Let $\rho = f(\theta)$ be the equation of a curve in polar coördinates. We desire to find an expression for the area bounded by the curve and the radii corresponding to the values $\theta = \theta_1$ and $\theta = \theta_2$. In Fig. 89, this area is bounded by the arc PQ and the radii OP and OQ. It will be referred to as the sector POQ.

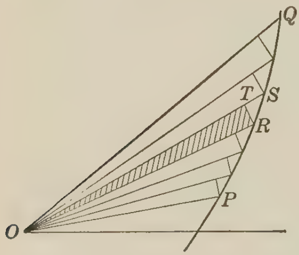

FIG. 89

Divide the angle POQ into n equal parts, calling each subdivision $\Delta\theta$ and the corresponding small sector ΔA. The required area equals $\Sigma\Delta A$, the sum of the n small sectors. Corresponding to each of these small sectors, construct a circular sector by drawing the arc of a circle with O as a center and the radius at the right-hand extremity of the sector as radius. Thus corresponding to the sector ROS we have the circular sector ROT, whose area is $\frac{1}{2}\rho^2\Delta\theta$, where $\rho = OR$. It can be shown that

$$\mathop{L}_{n\to\infty} \frac{\Delta A}{\frac{1}{2}\rho^2\Delta\theta} = 1.$$

Hence, by Duhamel's theorem (§ 88), we have the following expression for the area

$$A = \int_0^A dA = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta.$$

The *differential of area* in polar coördinates is $dA = \frac{1}{2}\rho^2 d\theta$. It differs from ΔA by an infinitesimal of higher order than $\Delta\theta$ (§ 30). It is known also as the *element of area* in polar coördinates.

Ex. 1. Find the area of the loop of the lemniscate $\rho^2 = a^2 \sin 2\theta$.

As a loop of the curve is obtained when θ runs through the range of values from 0 to $\frac{\pi}{2}$, we have

$$A = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta = -\frac{a^2}{4}\cos 2\theta\Big]_0^{\frac{\pi}{2}} = \frac{a^2}{2}.$$

Ex. 2. Find the area bounded by the circle $\rho = 2a\cos\theta$ and the cardioid $\rho = 2a(1 - \cos\theta)$.

The points of intersection of these curves are evidently $\left(a, \frac{\pi}{3}\right)$ and $\left(a, -\frac{\pi}{3}\right)$. From Fig. 90 we see that the area consists of two equal parts, of which one is $OAPBO$. Moreover it will be convenient to divide $OAPBO$ into two parts, $OAPO$ and $OPBO$.

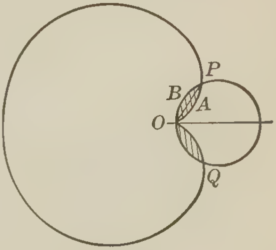

FIG. 90

$$\text{Area } OAPO = 2a^2\int_0^{\frac{\pi}{3}}(1-\cos\theta)^2 d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{3}}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{3}}(\tfrac{3}{2}-2\cos\theta+\tfrac{1}{2}\cos 2\theta)d\theta$$

$$= 2a^2\left(\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right)=a^2\left(\pi-\frac{7\sqrt{3}}{4}\right).$$

$$\text{Area } OPBO = 2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\, d\theta = a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta$$

$$= a^2\left(\frac{\pi}{2}-\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)=a^2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right).$$

$$\therefore \text{ Area } OAPBO = a^2\left(\frac{7\pi}{6}-2\sqrt{3}\right).$$

$$\text{Required area} = a^2\left(\frac{7\pi}{3}-4\sqrt{3}\right).$$

EXERCISES

1. Find the area of the circle $\rho = a$.

2. Find the area of the part of the circle $\rho = a\sin\theta$ lying in the first quadrant.

3. Find the total area of the lemniscate $\rho^2 = a^2\cos 2\theta$.

4. Find the total area of the cardioid $\rho = a(1+\cos\theta)$.

5. Find the area of the part of the parabola $\rho = \dfrac{2a}{1+\cos\theta}$ lying in the first quadrant.

6. Find the area swept over by the radius vector of the parabola $\rho = a\sec^2\dfrac{\theta}{2}$ as θ varies from 0 to $\dfrac{\pi}{2}$; from $\dfrac{\pi}{3}$ to $\dfrac{2\pi}{3}$.

7. Find the area of one loop of the curve $\rho = a\sin 3\theta$.

8. Find the area of one loop of the curve $\rho = a\sin n\theta$.

9. Find the total area of the curve $\rho = a \sin^3 \theta$.

10. Find the total area of the curve $\rho = a \sin^2 \theta$.

11. Find the area swept over by the radius vector of the spiral $\rho = e^\theta$ in one revolution, starting with $\theta = 0$.

12. Show that the area of a sector of the hyperbolic spiral $\rho\theta = a$ is proportional to the difference of the lengths of the bounding radii of the sector.

95. Length of arc. Let $y = f(x)$, or $x = \phi(y)$, be the equation of a curve in rectangular coördinates. We desire to find an expression for the length of arc of the curve from the point (x_1, y_1) to the point (x_2, y_2).

In Fig. 91 this arc is AB. Draw the ordinates at A and B, and divide the interval CD on the axis of x into n equal parts. At each of the points of subdivision draw the ordinate to the curve. This divides the arc into n parts, not equal in general. Call each of the parts Δs. The required length is $\Sigma \Delta s$.

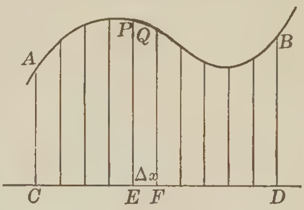

FIG. 91

In § 71 we saw that the differential of arc is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which says that

$$\underset{n \to \infty}{L} \frac{\Delta s}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x} = 1.$$

Hence by Duhamel's theorem (§ 88) we see, as in § 72, that the expression for the length of arc from (x_1, y_1) to (x_2, y_2) is

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (1)$$

At times it is more convenient to use y as the independent variable. Then, in an analogous manner, the expression for the length of arc is seen to be

$$s = \int_{y_1}^{y_2} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy. \tag{2}$$

If the equation of the curve is given in parametric form $x = f(t)$, $y = \phi(t)$, we replace dx and dy in $\sqrt{dx^2 + dy^2}$ by their values $dx = x'\,dt$ and $dy = y'\,dt$, where $x' = f'(t)$ and $y' = \phi'(t)$. Thus

$$ds = \sqrt{x'^2 + y'^2}\, dt.$$

$$\therefore s = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\, dt, \tag{3}$$

where t_1 and t_2 are the values of the parameter t at A and B respectively.

If the equation of the curve is given in polar coördinates, the differential of arc is $ds = \sqrt{\rho^2\, d\theta^2 + d\rho^2}$, as we saw in § 73; and the length of arc along the curve $\rho = f(\theta)$ from (ρ_1, θ_1) to (ρ_2, θ_2) is given by

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta, \tag{4}$$

when θ is taken as the independent variable, and by

$$s = \int_{\rho_1}^{\rho_2} \sqrt{\rho^2\left(\frac{d\theta}{d\rho}\right)^2 + 1}\, d\rho, \tag{5}$$

when ρ is taken as the independent variable.

EXERCISES

1. Find the length of the circumference of the circle

$$x^2 + y^2 - ax = 0.$$

2. Find the length of the arc of $x = \log \sec y$ between $y = 0$ and $y = \frac{\pi}{4}$.

3. Find the length of the arc of the curve

$$x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t)$$

from $t = 0$ to $t = 2\pi$.

4. Find the length of the arc of the curve

$$x = e^{-at} \cos \mu t, \quad y = e^{-at} \sin \mu t$$

from $t = 0$ to $t = 2\pi$.

5. Find the length of the epicycloid

$$x = (a + b) \cos \theta - a \cos \frac{a + b}{a} \theta, \quad y = (a + b) \sin \theta - a \sin \frac{a + b}{a} \theta$$

from cusp to cusp.

6. Find the length of the circumference of the circle $\rho = a$.

7. Find the length of that part of the circumference of the circle $\rho = a \cos \theta$ which lies in the first quadrant.

8. Find the length of a loop of the curve $\rho = a \sin^2 \theta$.

9. Find the length of the arc of the parabola $\rho = \dfrac{2a}{1 - \cos \theta}$ from the vertex to one end of the latus rectum.

10. Find the length of the cardioid $\rho = a(1 + \cos \theta)$.

11. Find the length of the entire curve $\rho = a \sin^3\left(\dfrac{\theta}{3}\right)$.

12. Find the length of the entire curve $\rho = a \sin^4\left(\dfrac{\theta}{4}\right)$.

13. Find the length of the entire curve $\rho = a \sin^n\left(\dfrac{\theta}{n}\right)$, for n a positive integer.

14. Find the length of the cissoid $\rho = 2a \tan \theta \sin \theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{4}$.

96. Volume of a solid of revolution. Let AB be an arc of the curve $y = f(x)$ which lies entirely on one side of the axis of x; and let $A = (a_1, b_1)$, $B = (a_2, b_2)$.

If AB is revolved about the axis of x, it generates a *surface of revolution*, and the area under it, $CABD$, generates a *solid of revolution.* In the next section we shall find an expression for the area of the surface of revolution. We shall consider here the problem of finding the volume of the solid of revolution.

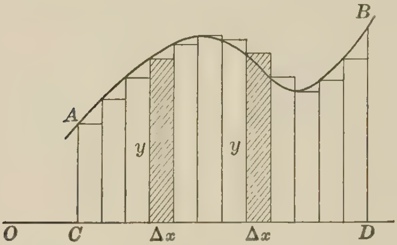

FIG. 92

Divide the interval CD into n equal parts, calling each part Δx. At each point of subdivision erect the ordinate to the curve, thus dividing the area under AB into n parts. Each of these generates a solid of revolution which we shall call ΔV. The required volume is $\Sigma \Delta V$, the sum of these n partial volumes.

Instead of each of the n partial areas under the curve, consider the rectangle which has Δx for base and the ordinate at its left-hand extremity for altitude. The solid generated by each of these rectangles is a cylinder, the radius of whose base is y, the altitude of the rectangle, and whose altitude is Δx. Hence the volume of this cylinder is $\pi y^2 \Delta x$.

It can be shown that

$$\underset{n \to \infty}{L} \frac{\Delta V}{\pi y^2 \Delta x} = 1.$$

Hence, by Duhamel's theorem (§ 88), we have for the required volume

$$V = \int_0^V dV = \pi \int_{a_1}^{a_2} y^2 \, dx. \qquad (1)$$

The *differential of volume of revolution is* $dV = \pi y^2 \, dx$. It is known also as the *element of volume of revolution.*

If the curve generating the surface of revolution is such that a vertical line cuts it in two points, in general, the volume generated by the area of the curve is evidently

$$V = \pi\int_{a_1}^{a_2}(y_2^2 - y_1^2)\,dx, \tag{2}$$

where y_2 is the larger of the two values of y corresponding to x, and y_1 is the smaller of the two. This volume is evidently the difference between the volumes generated by the areas *CAEBD* and *CAFGD*, Fig. 93.

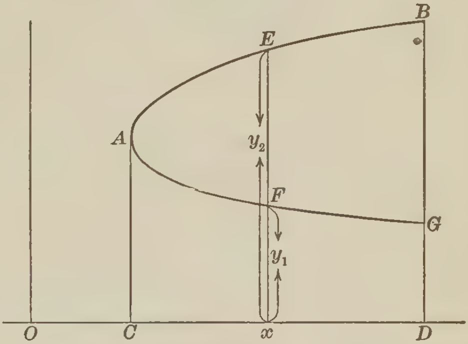

FIG. 93

If the curve generating the surface of revolution is such that some or all the ordinates corresponding to the various values of x in the interval (a_1, a_2) meet it in more than two points, a corresponding more or less obvious modification of the formula for each case, must be made.

If the curve is revolved about the axis of y, the volume of the solid generated is

$$V = \pi\int_{b_1}^{b_2}x^2\,dy, \tag{3}$$

if all the lines $y = b$, where $b_1 \leq b \leq b_2$, cut the curve in one point, and

$$V = \pi\int_{b_1}^{b_2}(x_2^2 - x_1^2)\,dy, \tag{4}$$

if the curve is cut twice by a horizontal line. Here x_2 and x_1 are respectively the larger and smaller abscissas of the two points of the curve having the same ordinate.

The formulas of this section are also applicable when the equation of the revolving curve is given in parametric form or in polar coördinates.

These cases will be considered in some of the illustrative exercises.

Ex. 1. Find the volume of a sphere considered as generated by revolving a semicircle about its diameter.

If the equation of the circle is $x^2 + y^2 = a^2$, that of the semicircle is $y = +\sqrt{a^2 - x^2}$. Hence the volume generated is

$$V = \pi \int_{-a}^{a} (a^2 - x^2)\, dx = \pi\left(a^2 x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = \frac{4}{3}\pi a^3.$$

Ex. 2. Find the volume generated by revolving a circle of radius a about a line in its plane at a distance b from its center.

If the line is taken as the axis of x, the equation of the circle is $x^2 + (y - b)^2 = a^2$. Solving for y, we have the two values $y_1 = b - \sqrt{a^2 - x^2}$, $y_2 = b + \sqrt{a^2 - x^2}$. Hence the volume generated is

$$V = \pi \int_{-a}^{a} (y_2^2 - y_1^2)\, dx = 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 2\pi^2 a^2 b.$$

When $b > a$, the solid generated is a ring, known as a *torus;*

when $b = a$, the volume generated equals $2\pi^2 a^3$.

It will be left as an exercise to show that when $b < a$, the line about which the circle revolves is a chord of the circle, and the value of V obtained is the difference between the volumes generated by the larger and smaller segments of the circle. What happens when $b = 0$?

Ex. 3. Find the volume generated by revolving the right-hand half of the ellipse $x = a \cos t$, $y = b \sin t$ about the y-axis.

Using formula (3),

$$V=\pi\int_{-b}^{b}x^2dy=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^2\cos^2 t\cdot b\cos t\,dt=2\,\pi a^2b\int_0^{\frac{\pi}{2}}\cos^3 t\,dt=\tfrac{4}{3}\,\pi a^2b.$$

Ex. 4. Find the volume generated by revolving the circle $\rho=2\,a\cos\theta$ about the y-axis.

Using the relations between rectangular and polar coördinates,

$$x=\rho\cos\theta,\ y=\rho\sin\theta,$$

for the given circle

$$x=2\,a\cos^2\theta,\ y=2\,a\sin\theta\cos\theta.$$

Putting these in formula (3),

$$V=\pi\int_{-a}^{a}x^2\,dy=\pi\int_{-a}^{a}4\,a^2\cos^4\theta\,d\,(2\,a\sin\theta\cos\theta)$$

$$=8\,\pi a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,(2\cos^2\theta-1)\,d\theta$$

$$=16\,\pi a^3\int_0^{\frac{\pi}{2}}(2\cos^6\theta-\cos^4\theta)\,d\theta.$$

Using Wallis' formula, we have at once

$V=16\,\pi a^3\left(\frac{2\cdot 5\cdot 3}{6\cdot 4\cdot 2}-\frac{3}{4\cdot 2}\right)\frac{\pi}{2}=2\,\pi^2a^3.$ [Compare case $b=a$, Ex. 2.]

It is not necessary to use formula (4) in this case. For as θ grows from $-\frac{\pi}{2}$ to $-\frac{\pi}{4}$, the arc OA, Fig. 94, is generated, and for it dy is negative. As θ grows from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, the arc ABC is generated, and for it dy is positive. Finally as θ grows from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, the arc CO is generated and for it dy is negative.

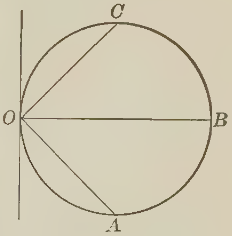

Fig. 94

Thus the use of formula (3) supplies the proper algebraic signs to the various elements of volume making up the required solid.

EXERCISES

1. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about a line in its plane parallel to the major axis and at a distance c from it.

2. Find the volume of a right circular cone by integration.

3. Find the volumes generated by revolving the area under the upper half of the part of the parabola $y^2 = 4ax$ to the left of the latus rectum about the x-axis and about the y-axis.

4. Find the volumes generated by revolving the area under cubical parabola $y = x^3$ from $x = 0$ to $x = 2$ about the x-axis and about the y-axis.

5. Find the volume generated by revolving about the y-axis the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes.

6. Find the volume generated by revolving about the x-axis the area under the upper half of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

7. Find the volume generated by revolving about the x-axis the area under an arch of the cycloid

$$x = a(t - \sin t), \ y = a(1 - \cos t).$$

8. Find the volume generated by revolving about the y-axis the area under an arch of the cycloid passing through the origin.

9. Find the volume generated by revolving the area under one half an arch of the cycloid about the normal at its highest point (its vertex).

10. Find the volume generated by revolving the area under an arch of the cycloid about the tangent at its vertex.

11. Find the volume generated by revolving about the x-axis the area under the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $x = -a$ to $x = a$.

12. Find the volume generated by revolving about the x-axis the area under the hyperbola $xy = 2$ lying to the right of $x = 2$.

13. Find the volume generated by revolving about the x-axis the area under the sine curve $y = \sin x$ from $x = 0$ to $x = \pi$.

14. Find the volume generated by revolving about the x-axis the area under the upper half of the cardioid

$$\rho = 2a(1 - \cos\theta).$$

15. Find the volume generated by revolving about the x-axis the area of a loop of the curve $\rho = a \sin 2\theta$.

16. Find the volume generated by revolving a loop of the curve $\rho = a \cos 2\theta$ about the y-axis.

17. Find the volume generated by revolving about the x-axis the area under the arc of the curve $\rho = a \sin\frac{\theta}{2}$ from $\theta = 0$ to $\theta = \pi$.

18. Find the volume generated by revolving about the x-axis the entire area under $y = \frac{8a^3}{x^2 + 4a^2}$.

19. Find the volumes generated by the revolution of the area of the part of $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ lying in the first quadrant, about the x-axis and the y-axis.

20. Find the volume generated by revolving the cissoid $y^2 = \frac{x^3}{2a - x}$ about its asymptote $x = 2a$.

21. Deduce the formula for the volume of the segment of a sphere, (*a*) of one base; (*b*) of two bases.

97. Area of surface of revolution. Let AB be an arc of the curve $y=f(x)$, where $A=(a_1, b_1)$ and $B=(a_2, b_2)$. We shall be interested in finding an expression for the area of the surface of revolution obtained by revolving AB about the x-axis. We shall suppose that AB lies entirely on one side of the axis.

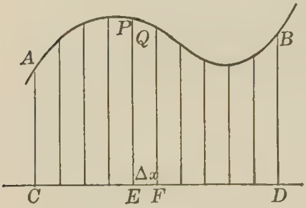

FIG. 95

Divide the interval CD into n equal parts, calling each part Δx. At each point of subdivision erect the ordinate to the curve, thus dividing the arc AB into n parts. Each of these generates a surface of revolution, which we shall call ΔS. The required surface is $\Sigma\Delta S$, the sum of these n partial surfaces.

Instead of each of the partial arcs of AB, consider the corresponding chord. In Fig. 95, corresponding to the partial arc PQ we have the chord $\overline{PQ}$. As AB revolves about the axis of x, $\overline{PQ}$ generates the frustum of a circular cone, whose altitude is $EF\equiv\Delta x$, and the radii of whose bases are $PE\equiv y$ and $QF\equiv y+\Delta y$ respectively.

The lateral area of this frustum is

$$\Delta F_S=\pi(PE+QF)\,PQ=\pi(2\,y+\Delta y)\sqrt{\Delta x^2+\Delta y^2}.$$

Since $\sqrt{\Delta x^2+\Delta y^2}$ differs from ds by an infinitesimal of higher order than Δx, ΔF_S differs likewise from $2\,\pi y\,ds$. Hence, by Duhamel's theorem (§ 88),

$$S=\underset{n\to\infty}{L}\Delta F_S=\int_{s_1}^{s_2}2\,\pi y\,ds. \qquad (1)$$

This will be taken as the definition of the area of the surface of revolution. The *differential of area of surface of revolution* is $dS=2\,\pi y\,ds$. The expression $2\,\pi y\,ds$ is known also as the *element of surface of revolution.*

In formula (1) there still remains the choice of taking x or y as the independent variable. This choice will be determined by the relative difficulty in evaluating the definite integrals

$$2\pi\int_{a_1}^{a_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx, \qquad 2\pi\int_{b_1}^{b_2} y\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy. \tag{1'}$$

If a vertical line $x=a$ cuts the curve in two or more points (a, y), (a, y_1), $\cdots$, (a, y_n), the area of the surface generated is evidently

$$S=2\pi\int_{s_1}^{s_2}(y_1+y_2+\cdots+y_n)\,ds. \tag{2}$$

If the curve is revolved about the axis of y, we have, in an analogous manner,

$$S=2\pi\int_{s_1}^{s_2} x\,ds, \tag{3}$$

where the choice of the independent variable is still at our disposal; and if a horizontal line cuts the curve in two or more points,

$$S=2\pi\int_{s_1}^{s_2}(x_1+x_2+\cdots+x_n)\,ds. \tag{4}$$

The formulas of this section are applicable also when the equation of the curve is given parametrically or in polar coördinates.

Ex. 1. Find the area of the surface generated by revolving a circle of radius a about a line in its plane at a distance b from its center.

If the line is taken as the axis of x, the equation of the circle is $x^2+(y-b)^2=a^2$. Solving for y, we have the two values $y_1=b-\sqrt{a^2-x^2}$, $y_2=b+\sqrt{a^2-x^2}$. Hence the area of the surface generated is given by

$$S=2\pi\int_{s_1}^{s_2}(y_1+y_2)\,ds=2\pi\int_{s_1}^{s_2}2\,b\,ds=4\,\pi b\int_{s_1}^{s_2}ds. \tag{5}$$

From the equation of the circle, $\frac{dy}{dx} = -\frac{x}{y-b}$.

$$\therefore\ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \frac{a}{y-b}\,dx = \frac{a\,dx}{\sqrt{a^2 - x^2}},$$

where the positive sign is to be taken with the radical since ds is positive.

$$\therefore\ S = 4\,\pi ab \int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}} = 4\,\pi ab \sin^{-1}\frac{x}{a}\bigg]_{-a}^{a} = 4\,\pi^2 ab.$$

Making use of the fact that the length of a semicircle is πa, the value of S as given by (5) may be obtained at once by noting that $s_1 = 0$ and $s_2 = \pi a$.

Ex. 2. Find the area of the surface generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the y-axis.

Here $$ds = \sqrt{dx^2 + dy^2} = 3\,a \sin t \cos t\, dt.$$

Using formula (3) and noting that the entire surface is obtained by letting t vary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we have

$$S = 2\,\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos^3 t \cdot 3\,a \sin t \cos t\, dt = 6\,\pi a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t \sin t\, dt.$$

Since $\cos^4 t \sin t$ is an odd function, this definite integral has the value zero. The reason is that the expression we have used for ds is negative for $-\frac{\pi}{2} \geq t < 0$ and in our formula for area of surface ds is supposed to be positive.[1] So we must write

$$S = -\,6\,\pi a^2 \int_{-\frac{\pi}{2}}^{0} \cos^4 t \sin t\, dt + 6\,\pi a^2 \int_{0}^{\frac{\pi}{2}} \cos^4 t \sin t\, dt.$$

---

[1] Because $ds^2 = 9\,a^2 \cos^2 t \sin^2 t\, dt^2$ is a perfect square, we are not justified in taking $ds = 3\,a \cos t \sin t\, dt$ for all values of t. For t in the fourth quadrant, $-\,3\,a \cos t \sin t$ is positive.

Letting

$-t$ be a new variable u, $-\int_{-\frac{\pi}{2}}^{0}\cos^4 t\sin t\,dt=\int_0^{\frac{\pi}{2}}\cos^4 u\sin u\,du.$

$$\therefore S=12\pi a^2\int_0^{\frac{\pi}{2}}\cos^4 t\sin t\,dt=-\frac{12\pi a^2}{5}\cos^5 t\Big]_0^{\frac{\pi}{2}}=\frac{12\pi a^2}{5}.$$

The fact that the area generated by the part of the curve determined by $-\frac{\pi}{2}\leq t\leq 0$ is equal to the area generated by the part determined by $0\leq t\leq\frac{\pi}{2}$ is obvious geometrically also.

Ex. 3. Find the area generated by revolving one half of the loop of the lemniscate $\rho^2=a^2\sin 2\theta$ about the line $\theta=\frac{\pi}{4}$.

Taking the line $\theta=\frac{\pi}{4}$ for initial line, we introduce a new set of polar coördinates determined by the equations of transformation

$$\rho=\rho,\quad \theta=\theta'+\frac{\pi}{4}.$$

In the new coördinates (ρ, θ'), the equation of the lemniscate takes the form

$$\rho^2=a^2\cos 2\theta'.$$

Dropping accents, our problem is now to find the area generated by one half of the loop of the lemniscate $\rho^2=a^2\cos 2\theta$ revolved about the initial line.

Using formula (1) we must express the quantities entering there in terms of ρ and θ, and then finally in terms of one of these, making use of the equation of the lemniscate. Thus we have

$$\rho=a\sqrt{\cos 2\theta},\quad \frac{d\rho}{d\theta}=-\frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}.$$

$$\therefore y=\rho\sin\theta=a\sin\theta\sqrt{\cos 2\theta},$$

$$ds=\sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}d\theta=\frac{a\,d\theta}{\sqrt{\cos 2\theta}}.$$

Putting these in (1),

$$S = 2\,\pi\,a^2 \int_0^{\frac{\pi}{4}} \sin\theta\,d\theta = -\,2\,\pi\,a^2 \cos\theta \Big]_0^{\frac{\pi}{4}} = \pi\,a^2(2 - \sqrt{2}).$$

EXERCISES

1. Find the area of the surface of a sphere of radius a. Use rectangular equation $x^2 + y^2 = a^2$ and parametric equations $x = a\cos\theta$, $y = a\sin\theta$, for the generating circle.

2. Find the area of a zone of altitude h on a sphere of radius a.

3. Find the area generated by revolving one half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis; also about its minor axis.

4. Find the areas required in Ex. 3, when the ellipse is given by its parametric equations $x = a\cos t$, $y = b\sin t$.

5. Find the lateral area of a right circular cone whose altitude is h and the radius of whose base is a.

6. Find the area generated by revolving the part of the parabola $y^2 = 4\,ax$ to the left of its latus rectum about the latus rectum.

7. Find the area generated by revolving about the x-axis the arc of the cubic parabola $3\,y = x^3$, between $x = 0$ and $x = 3$.

8. Find the area generated by revolving about the x-axis one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

9. Find the area generated by revolving about the y-axis an arch of the cycloid passing through the origin.

10. Find the area generated by revolving one half of an arch of the cycloid about the normal at its vertex.

11. Find the area generated by revolving an arch of the cycloid about the tangent at its vertex.

12. Find the area generated by revolving about the x-axis the arc of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

from $x = -a$ to $x = a$.

13. Find the area generated by revolving about the x-axis the upper half of the arc of the parabola $y^2 = 4\,ax$ extending from the origin to $x = b$.

14. Find the area of the reflecting surface of a parabolic mirror (a paraboloid of revolution) 16 inches deep and 48 inches wide.

15. Find the area generated by revolving about the x-axis the arc of the sine curve $y = \sin x$ from $x = 0$ to $x = \pi$.

16. Find the area generated by revolving about the x-axis the arc of the cardioid $\rho = 2\,a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = \pi$.

17. Find the area generated by revolving a loop of the lemniscate $\rho^2 = a^2 \cos 2\,\theta$ about the y-axis.

18. Find the area generated by revolving the part of the curve

$$x = e^t \cos t, \quad y = e^t \sin t,$$

from $t = 0$ to $t = \frac{\pi}{2}$, about the x-axis; also about the y-axis.

98. Volume of a solid with section of known area. If a solid is of such a shape that sections of it by planes parallel to one of the coördinate planes have known areas, the volume of the solid may be obtained by the following considerations. To fix the ideas, suppose that planes parallel to the yz-plane cut out sections of the solid, whose areas are known, and that the solid lies between the planes $x = a_1$ and $x = a_2$.

Divide the segment AB of the x-axis into n equal parts, letting each part be Δx. At each point of subdivision pass a plane perpendicular to the x-axis. Each plane cuts out a section, whose area $A(x)$ is known in terms of x, the distance of the cutting plane from the origin. If the slice of the solid between two consecutive cutting planes has the volume ΔV, the volume of the solid is equal to $\Sigma \Delta V$, the sum of all the slices. Consider now the set of cylinders, each having a section $A(x)$ for base and Δx for altitude. The volume of such a cylinder is $A(x)\,\Delta x$. It can be proved that

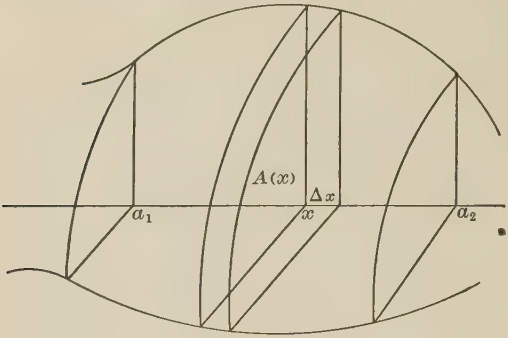

FIG. 96

$$\underset{n \to \infty}{L} \frac{\Delta V}{A(x)\,\Delta x} = 1.$$

Hence, by Duhamel's theorem (§ 88), we have for the volume of the solid

$$V = \int_0^V dV = \int_{a_1}^{a_2} A(x)\,dx. \tag{1}$$

The *differential of volume* is $dV = A(x)\,dx$; $A(x)\,dx$ being the *element of volume.*

The method of § 96 is a special case of that given here. For in the case of a solid of revolution about the x-axis, the curve cut out by a plane perpendicular to the x-axis is a circle whose area is πy^2.

In an analogous manner if the areas of sections of the solid by planes perpendicular to the y-axis have the known areas

$A(y)$, the volume between the planes $y = b_1$ and $y = b_2$ is given by

$$V = \int_{b_1}^{b_2} A(y)\,dy. \tag{2}$$

And similarly, if planes perpendicular to the z-axis cut out known areas $A(z)$, the volume between the planes $z = c_1$ and $z = c_2$ is

$$V = \int_{c_1}^{c_2} A(z)\,dz.$$

Ex. 1. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A plane perpendicular to the x-axis at the distance x from the origin cuts out the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

$$\text{or} \qquad \frac{y^2}{\frac{b^2}{a^2}\left(a^2 - x^2\right)} + \frac{z^2}{\frac{c^2}{a^2}\left(a^2 - x^2\right)} = 1.$$

Remembering that the area of the ellipse $\frac{y^2}{\alpha^2} + \frac{z^2}{\beta^2} = 1$ is $\pi\alpha\beta$, we see that the area of the section is $A(x) = \frac{\pi\,bc}{a^2}(a^2 - x^2)$. Hence

$$V = \frac{\pi bc}{a^2}\int_{-a}^{a} (a^2 - x^2)\,dx = \frac{4}{3}\pi\,abc.$$

Ex. 2. Find the volume of a wedge cut from the base of a right circular cylinder of base radius a by a plane through the diameter of the base and making an angle α with the plane of the base.

Let the axis of the cylinder be the axis of z and the diameter of the base lying in the cutting plane be the axis of y. A section

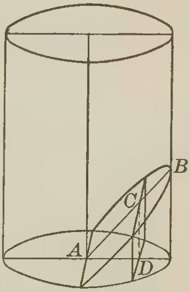

FIG. 97

by a plane perpendicular to the axis of x at a distance x from the origin is a rectangle whose altitude is $x \tan \alpha$ and whose base is $2\sqrt{a^2 - x^2}$. Hence $A(x) = 2x\sqrt{a^2 - x^2} \tan \alpha$,

and
$$V = 2 \tan \alpha \int_0^a x\sqrt{a^2 - x^2}\, dx = -\tfrac{2}{3} \tan \alpha\, (a^2 - x^2)^{\frac{3}{2}}\Big]_0^a$$
$$= \tfrac{2}{3}\, a^3 \tan \alpha.$$

EXERCISES

1. Remembering that the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes equals $\dfrac{a^2}{6}$, find the volume between the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate planes.

2. Remembering that the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ equals $\frac{3}{8}\pi a^2$, find the volume of the solid $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

3. Derive the formula for the volume of a cone whose altitude is h and the area of whose base is B.

4. Find the volume cut from $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2z$ by the plane $z = c$.

5. Find the volume of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^4}{c^4} = 1$.

6. Find the volume of the portion of the paraboloid of revolution $x^2 + y^2 = 8z^2$ lying inside the sphere $x^2 + y^2 + z^2 = 9$.

7. Find the volume of the conoid $x^2z^2 - b^2x^2 + a^2y^2 = 0$ included between the planes $x = 0$ and $x = c$.

8. Two wedges are cut from a right circular cylinder of base radius a and altitude h by planes through a diameter of one base and tangent to the other. Find the volume of the solid remaining.

9. Find the volume common to the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. [The solid common to the two cylinders is such that a section of it by the plane $x=x_0$ is a square whose side equals $2\sqrt{a^2-x_0^2}$.]

10. The cap of a gate post is the solid pictured in Fig. 98. CAE and DAB are two equal semicircles with their respective diameters CE and DB mutually perpendicular. The surfaces of the cap are right circular cylinders meeting in the semicircles CAE and DAB. Find the volume of the cap if $OD=a$.

11. Find the volume of the cap in Ex. 10 if the curves CAE and DAB are equal parabolas with $OD=a$ and $OA=b$.

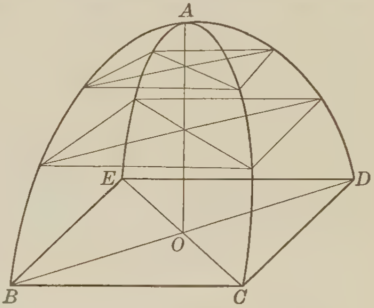

FIG. 98

12. Find the volume of the cap in Ex. 10 if the curves CAE and DAB are equal semi-ellipses with $OD=a$ and $OA=b$.

99. Work done by a force. A force is said to do work if the body acted upon by it changes its position. In the simplest case we have a constant force F with the point of application moving in the direction of the force. If the distance moved in this case is s, the work is given by the formula

$$W=Fs.$$

Thus the force required to raise a mass of 10 pounds being 10 pounds, if the body is raised 5 feet the work done by the force is 50 foot-pounds.

If the line of motion of the point of application does not coincide with the line of action of the force, only that component of the force which acts in the direction of the line of motion does work. If the angle between the two lines is α, the component of the force doing work is $F\cos\alpha$, and the work done is

$$W = Fs\cos\alpha.$$

Thus the work done in drawing a mass of 10 pounds up an inclined plane 5 feet long and making an angle of 60° with the vertical, neglecting friction, is

$$W = 50\cos 60° = 25 \text{ foot-pounds}.$$

If the mass is moved horizontally, no work is done at all, since now

$$W = 50\cos 90° = 0.$$

If the force is not constant, but varies continuously along the path of motion, the work done can be expressed as a definite integral. Let AB be the path of motion of the point of application, and let each point on this curve be determined by the value of s, its distance along the curve from some definite point O on it. Thus let A be the point s_1 and B the point s_2.

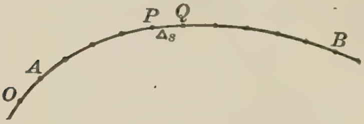

FIG. 99

Divide the path into n equal parts, each being designated by Δs. If the work done by the force over one of these parts of the path is designated by ΔW, the entire work is $\Sigma\Delta W$, the sum of all of these. When n is large the value of ΔW for the part PQ of AB, extending from s to $s + \Delta s$, is approximately $F\cos\alpha\,\Delta s$, where F is the value of the force at P and α is the angle between the tangent to the curve at P and the line of action of the force at that point; for F and α are nearly constant along PQ. We shall define $\underset{n\to\infty}{L}\sum F\cos\alpha\,\Delta s$ as the work done. Hence

$$W = \int_0^W dW = \int_{s_1}^{s_2} F\cos\alpha\, ds.$$

The *differential of work* is $dW = F \cos \alpha \, ds$; and $F \cos \alpha \, ds$ is known also as the *element of work*. To evaluate the integral both F and $\cos \alpha$ must be expressed as functions of s; or these, together with s, may be expressed as functions of some other convenient variable.

The following exercises will illustrate:

Ex. 1. Find the work done in stretching a helical spring from a length of 15 inches to 18 inches, if the length of the spring is 14 inches when no force is acting upon it, and if a force of 40 pounds is necessary to stretch it from 14 to 16 inches.

By Hooke's law, the amount of stretching of a helical spring is proportional to the force applied. If s is the number of inches it is stretched beyond its length when no force is acting, the force required is given by the formula

$$F = ks,$$

where k is a constant. In our exercise we have given

$$40 = 2\,k. \quad \therefore k = 20, \text{ and } F = 20\,s.$$

Moreover the line of action is in the direction of the force. Hence $\cos \alpha = 1$, and

$$\begin{aligned} W &= \int_1^4 F\,ds = 20 \int_1^4 s\,ds = 150 \text{ inch-pounds} \\ &= 12.5 \text{ foot-pounds.} \end{aligned}$$

Ex. 2. A quantity of gas is confined in a cylinder with a movable piston. Assuming that the law connecting p, the pressure of the gas on a unit area of the piston, and v, the volume of the gas, is $pv^k = c$, where c and k are constants, find the work done by the gas in pushing the piston, so that the altitude of the containing cylinder increases from s_1 to s_2.

If A is the cross section of the cylinder or the area of the piston, the force acting at any stage is Ap, and is in the direction of the motion. Hence

$$W = \int_{s_1}^{s_2} Ap\, ds.$$

In order to evaluate this, it is convenient to use v, the volume, as the variable. Evidently

$$As = v. \quad \therefore A\, ds = dv.$$

Hence we can write

$$W = \int_{v_1}^{v_2} p\, dv,$$

where $p = \dfrac{c}{v^k}$ and the initial and final volumes are $v_1 = As_1$, $v_2 = As_2$.

$$\therefore\ W = c\int_{v_1}^{v_2} \frac{dv}{v^k} = \frac{c}{k-1}\left(\frac{1}{v_1^{k-1}} - \frac{1}{v_2^{k-1}}\right) = \frac{p_1v_1 - p_2v_2}{k-1},$$

since $c = p_1v_1^k = p_2v_2^k$.

If the expansion takes place adiabatically, that is, without loss of heat, $k = 1.41$ approximately, in the case of air.

If the temperature is kept constant during the expansion, and the gas is approximately a perfect one, $k = 1$. In this case $pv = c$, and

$$W = c\int_{v_1}^{v_2} \frac{dv}{v} = c\,(\log v_2 - \log v_1) = c \log \frac{v_2}{v_1}.$$

Ex. 3. Two masses m_1 and m_2 are supposed concentrated at the points P_1 and P_2 respectively. The distance r, from P_1 to P_2, will be spoken of as the distance between the masses m_1 and m_2. If it is a initially, find the work done against the force of attraction in moving m_1 away from m_2 along some path, not necessarily the line P_1P_2, until its distance from m_2 is b.

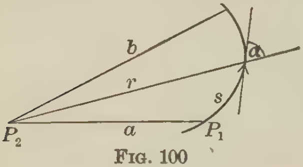

Fig. 100

The force of attraction between the two masses at any instant is

$$F = \frac{km_1m_2}{r^2},$$

where k is a constant. Since $\frac{dr}{ds} = \cos\alpha$ the work done is

$$W = \int_{s_1}^{s_2} F\cos\alpha\, ds = \int_a^b F\, dr = km_1m_2 \int_a^b \frac{dr}{r^2}$$
$$= km_1m_2\left(\frac{1}{a} - \frac{1}{b}\right).$$

The work done in this case is independent of the path.

Ex. 4. Find the work done in pumping out the water filling a hemispherical reservoir 20 feet deep.

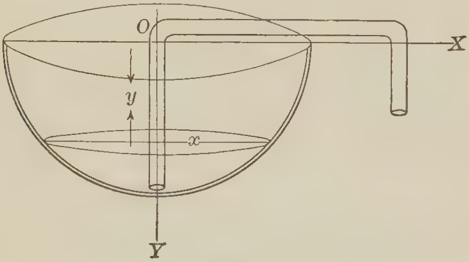

FIG. 101

Take the axes of coördinates as indicated in Fig. 101. An element of volume of water at a distance y below the surface is $\pi x^2\, dy = \pi(400 - y^2)\, dy$. If w is the weight of a cubic foot of water, the weight of this element of volume is $w\pi(400 - y^2)dy$, and the work done in raising this a height y is

$$w\pi y(400 - y^2)\, dy.$$

Hence the work done in emptying the reservoir is

$$W = w\pi\int_0^{20} y(400 - y^2)\, dy = w\pi\left[200\, y^2 - \frac{y^4}{4}\right]_0^{20} = 40{,}000\, w\pi.$$

Taking $w = 62.5$ pounds, $W = 2{,}500{,}000\,\pi$ foot-pounds.

EXERCISES

1. Find the work required to stretch a helical spring from a length of 8 inches to one of 12 inches if the spring when not acted upon is 6 inches long, and stretches 3 inches when acted upon by a force of 36 ounces.

2. Find the work necessary to compress a spring normally 12 inches long, from a length of 10 inches to one of 7 inches, if it requires a force of 56 pounds to compress it one inch. (Hooke's law applies in this case also.)

3. The force with which a spherical conductor, charged with a quantity e_1 of positive electricity, attracts a second spherical conductor, charged with a quantity e_2 of negative electricity, is

$$F=\frac{ke_1e_2}{r^2},$$

where k is a constant and r is the distance between the centers of the two conductors. If the force is 20 dynes when the centers are 50 centimeters apart, find the work done by the force of attraction when the distance between the centers changes from 100 centimeters to 75 centimeters.

4. A meteorite, 1,000,000 miles from the earth's center, falls upon the earth. The force of gravity varying inversely as the square of the distance of a body from the earth's center, find the work done by gravity, if the meteorite weighs w pounds at the earth's surface. Take the radius of the earth as 4000 miles.

5. When a body of weight w pounds falls s feet (a short distance, compared with the radius of the earth) near the earth's surface, the work done by gravity is ws foot-pounds. If a cubic foot of water weighs 62.5 pounds, how much work is done by gravity when the water in a cylindrical reservoir 60 feet in diameter and 16 feet high is drawn off through a pipe at the bottom?

HINT. The weight of an elementary slice is $900\,\pi\,62.5\,ds$ pounds.

6. If the reservoir is the frustum of a right circular cone 16 feet high, with the diameter of the upper base 200 feet and that of the lower base 144 feet, how much work is done by gravity when the water is discharged through a pipe at the bottom?

7. Van der Waal's equation for the expansion of a gas without change of temperature is

$$p = \frac{c}{v-b} - \frac{a}{v^2},$$

where a, b, c are constants. Find the work done when the gas expands from a volume v_1 to v_2.

8. The force required to stretch a bar of length L by an amount s is given by Hooke's law

$$F = \frac{EAs}{L},$$

where E is the coefficient of elasticity of the material of which the bar is made and A is the area of a cross section of the bar. Both are taken as constant. Hence the work of stretching the bar by an amount a is

$$W = \int_{s_1}^{s_2} F\,ds = \frac{EA}{L}\int_0^a s\,ds.$$

In the case of wrought iron, $E = 30{,}000{,}000$ pounds per square inch. What work is done in stretching a wrought iron bar having a cross section of 2.5 square inches from a length of 100 inches to 101 inches.

9. A conical reservoir 12 feet deep and 8 feet in diameter across the top is filled with oil weighing 50 pounds per cubic foot. Find the work done in pumping out the oil, if the pipe, through which the oil is pumped, is at the top of the reservoir.

10. Find the work done in pumping the water out of a cylindrical tank of base radius 4 feet and altitude 10 feet, if the pipe through which the water is pumped rises to a height of 10 feet above the tank.

11. Find the work done if in Ex. 10 the cylindrical tank is horizontal instead of standing on end.

12. Find the work done in dragging a body weighing 50 pounds along 60 degrees of arc of a vertical great circle on a sphere of radius 100 feet to the top of the sphere.

The force due to friction is proportional to the component of the force normal to the surface over which the body is being moved.

100. Liquid pressure. The pressure of a liquid upon a horizontal surface S of area A is equal to the weight of a column of the liquid having the area A for base and the distance from S to the top of the liquid for altitude. Thus if w is the weight of a unit volume of water and h is the depth at S, the pressure on S equals whA.

At any point P in the liquid there is a pressure exerted in every direction due to the action of gravity; and the pressure at P is numerically equal to the pressure of the liquid upon a unit horizontal area passing through P. Thus there is a pressure on the sides of a vessel containing a liquid. Suppose one of the sides of the vessel is vertical and plane. What is the pressure upon this vertical plane area?

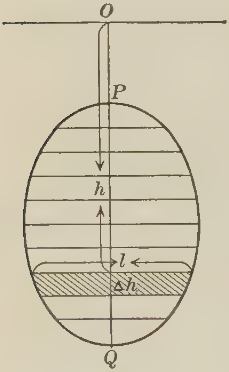

Fig. 102

Divide its greatest vertical length PQ into n equal parts, each equal to Δh. Through each point of subdivision draw a horizontal chord of the curve bounding the area. This divides the area into n parts. If Δp is the pressure on one of the parts, say the shaded part in Fig. 102, $\Sigma \Delta P$ equals the total pressure on the area.

If we replace the shaded area by $\Delta A = l \Delta h$, where l is the length of the chord forming the upper boundary of the shaded area, and h is the distance of this chord below the surface of the water, it can be shown that

$$\underset{n \to \infty}{L} \frac{\Delta p}{wh\Delta A} = 1.$$

Hence, by Duhamel's theorem (§ 88), we have for the pressure on the vertical side

$$p = w\int_{h_1}^{h_2} h\,dA = w\int_{h_1}^{h_2} hl\,dh, \qquad (1)$$

where l is the known function of h.

As before we shall take $w = 62.5$ pounds per cubic foot, in the case of water.

This method can be extended to determine the pressure upon surfaces which are neither plane nor vertical.

Ex. 1. One end of a water main, 6 feet in diameter, is closed by a vertical gate. Find the pressure on this gate if its center is 20 feet below the surface of the water.

Taking the center of the circular gate as the origin, the equation of the circle is $x^2 + y^2 = 9$. In formula (1), $l = 2x = 2\sqrt{9 - y^2}$, $h = 20 + y$, if y like h is measured vertically downward. $\therefore dh = dy$, and y ranges from -3 to 3. Hence we have for the pressure on the gate

$$p = 2w\int_{-3}^{3}(20 + y)\sqrt{9 - y^2}\,dy = 180\,\pi w = 11{,}250\,\pi \text{ pounds.}$$

Ex. 2. A hemispherical bowl $2a$ feet in diameter is full of water. What is the total pressure on the inner surface of the bowl?

In this case the formula for pressure is still

$$p = w\int_{h_1}^{h_2} h\,dA,$$

where dA is a horizontal elementary slice of surface and h is the distance of some point of this element below the surface of the water. Here we can take for element of area a zone of the surface of altitude dh. Then

$$dA = 2\,\pi a\,dh.$$

$$\therefore p = 2\,\pi a w \int_0^a h\,dh = 62.5\,\pi a^3 \text{ pounds.}$$

EXERCISES

1. Find the pressure on the side of a vertical rectangular floodgate 6 feet wide and 8 feet deep,

(*a*) if the upper edge lies in the surface of the water;

(*b*) if the upper edge is 10 feet below the surface of the water.

2. Find the pressure on the side of a vertical floodgate in the shape of an isosceles triangle whose base is 6 feet and whose altitude is 4 feet,

(*a*) if its base, the uppermost part of the gate, is in the surface of the water;

(*b*) if its vertex is 12 feet below the surface of the water.

3. Find the pressure on a vertical circular gate,

(*a*) if its diameter, 12 feet long, lies in the surface of the water;

(*b*) if its diameter is 16 feet below the surface of the water.

4. Find the pressure on a vertical semi-elliptical gate whose major and minor axes are 8 feet and 6 feet respectively, if the minor axis lies in the surface of the water.

5. Find the pressure on the side and the bottom of a cylindrical tank filled with water, if the radius of the base is b and the altitude is a.

6. Find the pressure on the side and the bottom of a cylindrical tank filled with half water and half oil which do not mix, if a cubic foot of oil weighs one half as much as a cubic

foot of water. Let the radius of the base of the tank be b and its altitude a.

7. Find the pressure on an inverted conical tank full of water, the altitude of the tank being a and the radius of the top b.

8. Find the pressure on the inverted conical tank if it is filled with one part (in bulk) of water and seven parts of oil which do not mix; the weight of the cubic foot of oil being $\frac{w}{2}$.

101. Mean value. Let $y = f(x)$ be a continuous function in the interval $a \leq x \leq b$. Divide this interval into n sub-intervals, each equal to $\Delta x = \frac{(b-a)}{n}$.

If $y_0 \equiv f(a), y_1, y_2, \cdots, y_{n-1}$ are the values of y corresponding to the values of x at the left-hand extremities of the successive subintervals, beginning to count at the left, the quotient

$$\frac{y_0 + y_1 + y_2 + \cdots + y_{n-1}}{n}$$

is the *arithmetic mean* of these n values of y. On replacing n by its value $\frac{(b-a)}{\Delta x}$, the arithmetic mean may be written

$$\frac{y_0 \Delta x + y_1 \Delta x + y_2 \Delta x + \cdots + y_{n-1} \Delta x}{b-a}.$$

The value of this expression depends upon n. As n approaches infinity, Δx approaches zero, and the arithmetic mean approaches the limit (§ 88)

$$\bar{y} = \frac{1}{b-a} \int_a^b f(x)\, dx, \tag{1}$$

which is known as the *mean value of* $y = f(x)$ *for the interval* (a, b). Compare theorem of the mean for integration, § 91.

The geometrical interpretation of the mean value of a function is very simple. Writing (1) in the form

$$\int_a^b f(x)\,dx = \bar{y}(b-a), \tag{2}$$

we note that the left-hand member of (2) is equal to the area under the curve from $x=a$ to $x=b$. The right-hand member is equal to the area of a rectangle whose base is $b-a$ and whose altitude is the mean value $\bar{y}$, which may also be called the *mean ordinate* of the curve.

Ex. 1. Find the mean ordinate in the first quadrant for the circle $x^2+y^2=a^2$, taking x as the independent variable.

Here $$\bar{y} = \frac{1}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi a}{4}.$$

Ex. 2. Find the mean ordinate in the first quadrant of the circle $x^2+y^2=a^2$, taking s, the length of arc of the circumference, as the independent variable.

Here $$y = a\sin\frac{s}{a}.$$

$$\therefore\ \bar{y} = \frac{2}{\pi a}\int_0^{\frac{\pi a}{2}} \sin\frac{s}{a}\,ds = \frac{2a}{\pi}.$$

Ex. 3. Find the mean velocity of a falling body, starting from rest, near the earth's surface, taking the time t as the independent variable; also the distance s as the independent variable.

Taking $v=gt$, $\bar{v} = \frac{1}{t_1}\int_0^{t_1} gt\,dt = \frac{1}{2}gt_1 = \frac{1}{2}v_1$, where t_1 is the interval of time during which the body falls, and v_1 is its final velocity.

Taking $v=\sqrt{2gs}$, $\bar{v} = \frac{1}{s_1}\int_0^{s_1}\sqrt{2gs}\,ds = \frac{2}{3}\sqrt{2gs_1} = \frac{2}{3}v_1$. Here s_1 is the distance the body falls in the time t_1.

EXERCISES

1. Find the mean ordinate of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$; also from $x = 3$ to $x = 12$.

2. Find the mean ordinate of $y = x(4 - x^2)$ from $x = 0$ to $x = 2$; also from $x = -2$ to $x = 2$.

3. Find the mean length of all the chords drawn from a point on the circumference of a circle of radius a, taking for independent variable the angle which each chord makes with the diameter through the common point.

Hint. Take the polar equation $\rho = 2a\cos\theta$ and find the mean value of ρ.

4. In the motion of a simple pendulum $s = a\cos\omega t$, a complete period is $\frac{2\pi}{\omega}$. What are the mean distance and the mean velocity in the interval of time $t = 0$ to $t = \frac{\pi}{2\omega}$? Also in the interval $t = 0$ to $t = \frac{\pi}{\omega}$? Explain.

5. Find the mean velocity of the simple pendulum during the first quarter period when the distance s is taken as the independent variable.

6. Find the mean velocity of a falling body near the earth's surface between the time $t = 2$ seconds and $t = 4$ seconds.

7. If it requires a force of 40 pounds to stretch a spring from its normal length of 10 inches to 12 inches, what is the average or mean force required to stretch it from 11 inches to 13 inches?

8. Since by Hooke's law the force required to stretch a spring a length s beyond its normal length is

$$F = ks,$$

show that the average or mean force required to stretch it from a length a beyond its normal length to a length b beyond its normal length is

$$\overline{F} = \tfrac{1}{2}k(a + b);$$

that is, the arithmetic mean of the forces required to stretch it the lengths a and b, respectively, beyond its normal length.

9. Show that if y is any linear function of x, the mean value of y is the arithmetic mean of the initial and final values of y.

10. The range of a projectile fired with an initial speed v_0 and an angle of elevation α is $\frac{v_0^2 \sin 2\alpha}{g}$, [(9), § 40]. Find the mean range as α varies from 0 to $\frac{\pi}{2}$.

102. Supplementary exercises. For purposes of review of this chapter, the following exercises are added, their order bearing no relation to that in which the topics, illustrated by them, were treated.

EXERCISES

1. Find the area under an arch of $y = \cos x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

2. Find the volume generated by revolving half an arch of $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{2}$, about the x-axis; about the y-axis.

3. Find the area of the surface generated by revolving an arch of $y = \cos x$ about the x-axis.

4. Find the area swept over by the radius vector of the spiral of Archimedes $\rho = a\theta$ as θ varies from 0 to 2π; from 2π to 4π; from $2(n-1)\pi$ to $2n\pi$.

5. Find the length of the spiral of Archimedes $\rho = a\theta$ from $\theta = 0$ to $\theta = \theta_1$.

6. Find the area of the loop of $x^2 = (y-1)(y-2)^2$.

7. Find the length of the arc of $y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$.

8. Find the area under $y = e^{-x}$ from $x = 0$ to $x = \infty$.

9. Find the volume generated by revolving about the x-axis the area under $y = e^{-x}$ from $x = 0$ to $x = \infty$.

10. Find the area of the surface generated by revolving about the x-axis the arc of $y = e^{-x}$ from $x = 0$ to $x = \infty$.

11. Find the volume generated by revolving about the x-axis the area under the arc of $y = \sec x$ from $x = 0$ to $x = \frac{\pi}{4}$.

12. By integration find the volume of the tetrahedron formed by the coördinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

13. Find the area of the cardioid $\rho = a(1 - \cos\theta)$.

14. Find the area of the cardioid $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$.

15. Find the area of the surface generated by revolving about the x-axis half the cardioid $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$.

16. A circular cone with its vertex in the surface of a sphere of radius a has a diameter of the sphere for axis. Find the volume common to the two solids. Let the vertical angle of the cone be 2α.

17. Two cylinders of equal altitude h have a common lower base of radius a. Their upper bases are tangent to each other. What is the volume common to the two cylinders?

18. Find the area of the entire curve $\rho = a(\sin 2\theta + \cos 2\theta)$. Sketch the curve.

19. Find the length of the entire curve $\rho = a\sin\theta + b\cos\theta$. Sketch the curve. In particular let $a = b = 1$.

20. Find the area swept over by the radius vector of $\rho = a\sin\theta + b\cos\theta$ as θ varies from 0 to $\frac{\pi}{2}$. Find the area of the entire curve.

21. Find the area of the loop of $3ay^2 = x(a-x)^2$.

22. Find the volume generated by revolving half the loop of $3ay^2 = x(a-x)^2$ about the x-axis.

23. Find the area of the surface generated by revolving half the loop of $3ay^2 = x(a-x)^2$ about the x-axis.

24. Find the volume common to $y^2 + z^2 = 4ax$ and $x^2 + y^2 = 2ax$.

25. Find the area under $y = \log x$ from $x = 1$ to $x = a$.

26. Find the area between $y = \log x$, $x = 0$, $y = 0$, and $y = 1$.

27. Find the area of one loop of $\rho = a \sin 3\theta$.

28. Find the entire length of the epicycloid

$$x = \frac{a}{2}(3\cos t - \cos 3t),\quad y = \frac{a}{2}(3\sin t - \sin 3t).$$

29. Find the area of the surface generated by revolving the arc of the epicycloid

$$x = \frac{a}{2}(3\cos t - \cos 3t),\quad y = \frac{a}{2}(3\sin t - \sin 3t)$$

from $t = 0$ to $t = \pi$, about the x-axis.

30. Find the area of the loop of $\rho^2 = a^2 \cos 2\theta \cos\theta$ which is bisected by the initial line.

31. Find the area of the surface generated by revolving the upper half of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

32. Find the length of the entire curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

33. Find the area of a loop of $8a^2y^2 = a^2x^2 - x^4$.

34. The plane of a moving circle is perpendicular to that of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the radius of the circle is an ordinate of the ellipse. Find the volume generated when the circle moves from one vertex of the ellipse to the other.

35. Find the area generated by the revolution, about its chord, of an arc of a circle subtending an angle $2\,\alpha$, the radius of the circle being a.

36. Find the area between the parabolas $y^2 = ax$ and $x^2 = by$.

37. Find the length of $\rho = a \cos^4 \frac{\theta}{4}$ from $\theta = 0$ to $\theta = 2\,\pi$.

38. Find the area of the surface generated by revolving about the y-axis the arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $x = 0$ to $x = a$.

39. Find the volume generated by revolving a loop of the lemniscate $\rho^2 = a^2 \cos 2\,\theta$ about the y-axis.

40. Find the length of $x = e^{-t} \cos t$, $y = e^{-t} \sin t$ from $t = 0$ to $t = \frac{\pi}{2}$.

41. Find the area between the two circles $\rho = a \cos \theta$ and $\rho = a \sin \theta$.

42. Show that the length of arc of the logarithmic spiral $\rho = e^{a\theta}$ between any two points is proportional to the difference of the radius vectors of the points.

43. Find the area of the loop of the folium of Descartes $\rho = \dfrac{3\,a \tan \theta \sec \theta}{1 + \tan^3 \theta}$.

44. Find the length of $y = \log \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

45. Show that the area under an arc of $xy = a^2$ from (x_1, y_1) to (x_2, y_2) is equal to the area bounded by that arc of the curve and the lines $x = 0$, $y = y_2$, and $y = y_1$.

46. Find the volume generated by revolving about the y-axis the area in the first quadrant bounded by the witch $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$, the y-axis, and the line $y = a$.

47. Find the total area of $\rho = 3 + 2 \cos 4 \theta$.

48. Find the volume cut from a sphere of radius a by a right circular cylinder with base radius $b < a$, whose axis passes through the center of the sphere.

49. Find the volume cut from a sphere of radius a by a right circular cylinder with base radius $\frac{a}{2}$ and an element through the center of the sphere.

50. Find the length of the arc of $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ from $t = 0$ to $t = 4\pi$.

51. Find the area between $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and $x + y = a$.

52. Find the length of $y = a \log \frac{a^2}{a^2 - x^2}$ from $x = 0$ to $x = \frac{a}{2}$.

53. Find the area between the parabolas $\rho = \frac{2a}{1 - \cos \theta}$ and $\rho = \frac{2a}{1 + \cos \theta}$.

54. A segment of a circle is revolved about a diameter parallel to its chord. Show that the volume generated is equal to that of a sphere whose diameter is equal to the chord.

55. Find the area of the surface generated by revolving the arc of $6a^2xy = x^4 + 3a^4$ from $x = a$ to $x = 2a$ about the x-axis; about the y-axis.

56. Find the volume of the portion of the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2c(z + h)$ below the plane $z = 0$.

57. Find the mean value of the ordinates of an arch of $y = \sin x$, x being the independent variable.

58. A horizontal cylindrical tank is half full of oil, weighing 50 pounds per cubic foot. The diameter of each end of the tank is 6 feet. Find the pressure on each end, and on the cylindrical part. What are the respective pressures when the tank is full of oil? [Let h be length of tank.]

59. A positive charge e of electricity is fixed at a point O. The repulsion of a unit positive charge at a distance x from O is $\frac{e}{x^2}$. Find the work done in bringing up a unit positive charge from ∞ to a point P distant a from O.

60. When an electric current flows a distance x through a homogeneous conductor of cross section A, the resistance is $\frac{kx}{A}$, where k is a constant depending upon the material of which the conductor is made. Find the resistance when the current flows from the inner to the outer surface of a hollow sphere, the respective radii being a and b. [Taking as an element of the conductor a spherical shell of thickness dr and inner radius r, the element of resistance is $\frac{kdr}{4\pi r^2}$.]

61. Find the resistance when the current flows from the inner to the outer surface of a hollow cylinder of length l, the radii of the two surfaces being a and b respectively.

62. Find the resistance when the current flows through a frustum of a cone of height h, the areas of the two bases being b_1 and b_2 respectively.

63. The kinetic energy of a body of mass m moving with a speed v is $\frac{mv^2}{2}$. If a body describes the simple harmonic motion $s = a \cos kt$, show that the mean kinetic energy during a complete vibration is one half the maximum kinetic energy if t is taken as the independent variable, and is two thirds the maximum kinetic energy if s is taken as the independent variable.

64. Find the pressure on a sphere two feet in diameter immersed in water, its center being 20 feet below the surface.

65. A spherical bag of radius 10 inches contains gas at a pressure of 18 pounds per square inch. Find the work done in compressing it into a sphere of radius 5 inches, the gas obeying the law $pv =$ constant.

The element of work done in compressing the bag at any stage when its radius is r to a neighboring stage when its radius is $r - dr$ is $pAdr$, where $A = 4\pi r^2$.

66. Find the mean pressure per square inch in Ex. 65 when v is the independent variable; when r is the independent variable.

67. Find the mean area of the plane sections of a right circular cone of altitude a and base radius b made by planes parallel to the base, the altitude being the independent variable.

68. Find the mean area of the plane sections of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ made by planes perpendicular to the x-axis.

CHAPTER XII

FUNCTIONS OF TWO OR MORE VARIABLES

103. Functions of several variables. Several variables are said to be *independent* if arbitrary independent values (or in certain cases arbitrary independent values within certain limits) may be assigned to each of them. A quantity is said to be a *function* of these independent variables, if it is determined whenever any set of values (within certain limits possibly) is assigned to the variables. If a single definite value of the function corresponds to each given set of values of the variables, the function is said to be *single-valued.* If any one of several values of the function corresponds to each given set of values of the variables, the function is said to be *many-valued.*

Thus if r is the radius of the base of a cylinder and h is its altitude, its volume is expressed by the formula $V = \pi r^2 h$. Here r and h may have any positive values whatever, and may therefore be taken as two independent variables. The quantity V is known when r and h are assigned. It is a function of the two variables r and h.

Again, in the equation of a sphere of radius a and with its center at the origin

$$x^2 + y^2 + z^2 = a^2, \tag{1}$$

we have three variables x, y, z connected by a single relation. Any two of them may be considered as independent, then the third will be a function of the other two. Thus, letting x and y be the independent variables,

$$z = \pm\sqrt{a^2 - x^2 - y^2}. \tag{2}$$

Here, in general, z may have one of two values, corresponding to a given pair of values of x and y. It is not a single-valued function as defined by (2). But if we restrict ourselves to the upper hemisphere by the definition

$$z = +\sqrt{a^2 - x^2 - y^2}, \tag{3}$$

then z is a single-valued function of x and y.

It should also be noted that if we restrict the values of the variables to correspond to real points on the sphere, x and y must be so chosen that

$$x^2 + y^2 \leqq a^2.$$

But this restriction still allows a wide range of choice of independent values for x and y.

At times we have occasion to consider functions of several variables, all of which are not independent. Thus in the formula for the area of a triangle in terms of its sides a, b, c,

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

S is given as a function of the four variables s, a, b, c. But these are not independent, being connected by the relation

$$a + b + c = 2\,s.$$

104. Partial derivative. Let the function[1]

$$u = f(x, y, z)$$

be single-valued. Corresponding to a given set of values of x, y, z, the function assumes a definite value u. Corresponding to a set of neighboring values $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, the function assumes a value which, in general, is different from u; let it be designated by

$$u + \Delta u = f(x + \Delta x,\ y + \Delta y,\ z + \Delta z). \tag{2}$$

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[1] A function of three independent variables has been selected as illustrative of the general case. What is given here can be applied readily to functions of two or more independent variables.

Thus, corresponding to the increments Δx, Δy, Δz of the variables, we have the increment of the function

$$\Delta u = f(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - f(x, y, z). \tag{3}$$

Here Δx, Δy, Δz are independent of one another, because x, y, z are supposed to be independent variables. Making each two of them zero in turn, we have the three partial increments of u

$$\left.\begin{aligned} \Delta_x u &= f(x+\Delta x,\ y,\ z) - f(x, y, z), \\ \Delta_y u &= f(x,\ y+\Delta y,\ z) - f(x, y, z), \\ \Delta_z u &= f(x,\ y,\ z+\Delta z) - f(x, y, z). \end{aligned}\right\} \tag{4}$$

Dividing each of these in turn by the increment of the corresponding variable, and passing to the limit indicated, we have

$$\underset{\Delta x \longrightarrow 0}{L} \frac{\Delta_x u}{\Delta x} = \frac{\partial u}{\partial x} \text{ or } u'_x, \qquad \underset{\Delta y \longrightarrow 0}{L} \frac{\Delta_y u}{\Delta y} = \frac{\partial u}{\partial y} \text{ or } u'_y,$$

$$\underset{\Delta z \longrightarrow 0}{L} \frac{\Delta_x u}{\Delta z} = \frac{\partial u}{\partial z} \text{ or } u'_z. \tag{5}$$ [1]

These are known as the *partial derivatives* of u with respect to x, y, and z respectively. Evidently, *the partial derivative of a function of several variables, with respect to one of them, is the derivative, with respect to that variable, of the function arising when all the other variables are assumed to be constant.*

Thus if $u = x^3 + y^3 + z^3 - 6\,xyz$,

$$\frac{\partial u}{\partial x} = 3(x^2 - 2\,yz), \quad \frac{\partial u}{\partial y} = 3(y^2 - 2\,xz), \quad \frac{\partial u}{\partial z} = 3(z^2 - 2\,xy).$$

If $u = \tan^{-1}\dfrac{y}{x}$, $\qquad u'_x = \dfrac{-\,y}{x^2+y^2}, \qquad u'_y = \dfrac{x}{x^2+y^2}.$

If $u = x^{yz}$, $\quad u'_x = x^{yz-1}yz, \quad u'_y = x^{yz}z\log x, \quad u'_z = x^{yz}y\log x.$

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[1] It is customary to represent partial differentiation by using a round "∂," or by affecting the symbol for the function with a prime to indicate differentiation and adding a subscript to indicate the variable with respect to which the derivative is taken.

Using the formula for the derivative of a function of a function of one variable (§ 11), it follows readily that if $u = f(v)$, where $v = \phi(x, y, z)$,

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial z} = f'(z)\frac{\partial v}{\partial z}, \tag{6}$$

where $f'(v)$ is the ordinary derivative $\frac{df}{dv}$, $f(v)$ being a function of the single variable v.

Of course, whenever the function $f(v)$ is known, the student naturally carries out the steps indicated in (6). But at times it is necessary to use the partial derivatives of functions of several variables, which are given symbolically. In such cases, formulas (6) will be found valuable. Thus if $u = f(xy) + \phi\left(\frac{y}{x}\right)$,

$$\frac{\partial u}{\partial x} = yf'(xy) - \frac{y}{x^2}\phi'\left(\frac{y}{x}\right), \quad \frac{\partial u}{\partial y} = xf'(xy) + \frac{1}{x}\phi'\left(\frac{y}{x}\right).$$

EXERCISES

Find the partial derivatives of the following ten functions:

1. $u = x^2y + xy^2$.
2. $u = x^2yz + xy^2z + xyz^2$.
3. $u = x \sin(y + z)$.
4. $u = \sqrt{x^2 + y^2 + z^2}$.
5. $u = \sin x \cos y + \cos x \sin y$.
6. $u = \log \frac{x^3 + y^3 + z^3}{xyz}$.
7. $u = \frac{x}{y + z}$.
8. $u = \sin^{-1}\frac{x}{y}$.
9. $u = f(x + y + z) - \phi(xyz)$.
10. $u = f(x^2 + y^2)\phi(xy)$.
11. If $u = x^2 + y^2 - 3xy$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$.
12. If $u = \sin\frac{x + y}{z}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

13. If $u = \tan^{-1}\frac{y}{x}$, show that $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = -1$.

14. If $u = \frac{x+y}{x-y}$, show that $yu'_x + xu'_y = 2u$.

15. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, show that $u'^2_x + u'^2_y + u'^2_z = u^4$.

16. If $u = \log(x^3 + x^2y + yz^2 + z^3)$, show that $xu'_x + yu'_y + zu'_z = 3$.

Remark. Extending the notion of a continuous function of a single variable (§ 5), we say that a function of several variables $u = f(x, y, z)$ is *continuous* at a set of values $x = a$, $y = b$, $z = c$ [or to use a geometrical form of expression we shall say it is continuous at the point (a, b, c)] if

$$\underset{x \to a,\, y \to b,\, z \to c}{L} f(x, y, z) = f(a, b, c).$$

This amounts to saying that $u = f(x, y, z)$ is continuous at any point (x, y, z), if the increment Δu is an infinitesimal whenever Δx, Δy, and Δz are.

105. Higher partial derivatives. The partial derivatives of a function of several variables are also functions of the variables. Differentiating these partially, we obtain what are known as *higher partial derivatives* of the original function. Convenient notations for these are the following:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \text{ or } u''_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \text{ or } u''_{xy}, \quad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial z\,\partial x} \text{ or } u''_{xz},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y} \text{ or } u''_{yx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} \text{ or } u''_{yy}, \quad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial z\,\partial y} \text{ or } u''_{yz},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x\,\partial z} \text{ or } x''_{zx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y\,\partial z} \text{ or } u''_{zy}, \quad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial z^2} \text{ or } u''_{zz}.$$

These are known as *second partial derivatives* of u, while $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are known as *first partial derivatives*.

Partial derivatives of the second partial derivatives are known as third partial derivatives; and so on. Thus

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)=\frac{\partial^3 u}{\partial x^3} \text{ or } u'''_{xxx},\ \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}\right)=\frac{\partial^3 u}{\partial y\,\partial x^2} \text{ or } u'''_{xxy},\ \cdots,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right)=\frac{\partial^3 u}{\partial x\,\partial y\,\partial x} \text{ or } u'''_{xyx},\ \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right)=\frac{\partial^3 u}{\partial y^2\,\partial x} \text{ or } u'''_{xyy},\ \cdots,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial z\,\partial y}\right)=\frac{\partial^3 u}{\partial x\,\partial z\,\partial y} \text{ or } u'''_{yzx},\ \cdots,$$

. . . .

Starting with the three partial derivatives of the first order in the case of a function of three independent variables, we see there are

$3^2 = 9$ partial derivatives of the second order,

$3^3 = 27$ partial derivatives of the third order,

and so on.

But these are not all distinct. It can be proved (the proof is beyond the scope of this book) that under certain conditions of continuity of the functions involved, the value of a higher partial derivative is independent of the order in which the successive differentiations with respect to the several variables are effected. Thus in the case of the types of function we are apt to meet, the following equalities hold:

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x},$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 u}{\partial x^2\,\partial y}=\frac{\partial^3 u}{\partial y\,\partial x^2},$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=\frac{\partial^3 u}{\partial y\,\partial x\,\partial z}=\frac{\partial^3 u}{\partial z\,\partial y\,\partial x}=\cdots,$$

and so on.

For example, if $u = x^4y^2 - xy^3$,

$$\frac{\partial u}{\partial x} = 4\,x^3y^2 - y^3, \qquad \frac{\partial u}{\partial y} = 2\,x^4y - 3\,xy^2,$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 8\,x^3y - 3\,y^2, \qquad \frac{\partial u}{\partial x\,\partial y} = 8\,x^3y - 3\,y^2,$$

$$\frac{\partial^3 y}{\partial x\,\partial y\,\partial x} = 24\,x^2y, \qquad \frac{\partial^3 u}{\partial x^2\,\partial y} = 24\,x^2y.$$

It will be left as an exercise to show that $\dfrac{\partial^3 u}{\partial y\,\partial x^2} = 24\,x^2y.$

Again, if $u = \sqrt{x^2 + y^2 + z^2}$,

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \qquad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \frac{\partial^2 u}{\partial z\,\partial x} = \frac{-\,xz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

$$\frac{\partial^2 u}{\partial z\,\partial y} = \frac{-\,yz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \qquad \frac{\partial^2 u}{\partial y\,\partial z} = \frac{-\,yz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

$$\frac{\partial^3 u}{\partial y\,\partial z\,\partial x} = \frac{3\,xyz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \qquad \frac{\partial^3 u}{\partial x\,\partial z\,\partial y} = \frac{3\,xyz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{3\,xyz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}.$$

Analogous to what was given in § 104 concerning first partial derivatives of a function of a function of several variables, the higher partial derivatives of such a function can be expressed as follows:

Given $u = f(v)$, where $v = \phi(x, y\ z)$, we have, as before

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial z} = f'(v)\frac{\partial v}{\partial z},$$

where $f'(v)$ is the derivative $\dfrac{df}{dv}$, $f(v)$ being a function of the single variable v.

Differentiating each of these as a product

$$\frac{\partial^2 u}{\partial x^2} = f''(v)\left(\frac{\partial v}{\partial x}\right)^2 + f'(v)\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial^2 v}{\partial y\,\partial x} = f''(v)\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + f'(v)\frac{\partial^2 v}{\partial y\,\partial x}, \cdots.$$

Similarly

$$\frac{\partial^3 v}{\partial x^3} = f'''(v)\left(\frac{\partial v}{\partial x}\right)^3 + 3\,f''(v)\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + f'(v)\frac{\partial^3 v}{\partial x^3},$$

$$\frac{\partial^3 v}{\partial y\,\partial x^2} = f'''(v)\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial v}{\partial y} + f''(v)\left[2\,\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y\,\partial x} + \frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y}\right]$$
$$+ f'(v)\frac{\partial^3 v}{\partial y\,\partial x^2},$$

$$\frac{\partial^3 v}{\partial z\,\partial y\,\partial x} = f'''(v)\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + f''(v)\left[\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial z\,\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z\,\partial x} + \frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\,\partial x}\right]$$
$$+ f'(v)\frac{\partial^3 v}{\partial z\,\partial y\,\partial x},$$

and so on.

Thus if $u = \phi(x + ay) + \psi(x - ay)$,

$$\frac{\partial u}{\partial x} = \phi'(x + ay) + \psi'(x - ay), \quad \frac{\partial u}{\partial y} = a\phi'(x + ay) - a\psi'(x - ay),$$

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x + ay) + \psi''(x - ay), \quad \frac{\partial^2 u}{\partial y^2} = a\phi''(x + ay) + a\psi''(x - ay).$$

Hence in particular

$$a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

EXERCISES

1. If $u = \sin(x - y)$, verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ and $u'''_{xyy} = u'''_{yxy} = u'''_{yyx}$.

2. If $u = (x + y)e^{y+z}$, verify $\frac{\partial^4 u}{\partial z^2\,\partial y\,\partial x} = \frac{\partial^4 u}{\partial y\,\partial z^2\,\partial x} = \frac{\partial^4 u}{\partial y\,\partial x\,\partial z^2}$.

3. If $u = e^{xyz}$, verify $u''_{xz} = u''_{zx}$, $u'''_{xxz} = u'''_{xzx}$, $u'''_{xyz} = u'''_{yzx}$.

4. If $u = \sin(xy)$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

5. If $u = \tan^{-1}\dfrac{y}{x}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

6. If $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that $u''_{xx} + u''_{yy} + u''_{zz} = 0$.

7. If $u = e^{-y}\,\phi(x - y)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y\,\partial x} + \dfrac{\partial u}{\partial x} = 0$.

8. If $u = x\phi\left(\dfrac{y}{x}\right) + (x - 1)y\log x$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} = (x + 1)y$.

9. If $u = z\phi(x^2 - y^2) + xyz$, show that $y^2u''_{xx} - x^2u''_{yy} + z^2u''_{zz} + yu''_{xz} + xu''_{yz} - xu'_x + yu'_y = x^2 + y^2$.

10. If $u = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, show that $x^2u''_{xx} + 2\,xyu''_{xy} + y^2u''_{y} = 0$.

11. If $u = x\phi(x + y) + y\psi(x + y)$, show that $u''_{xx} - 2\,u''_{xy} + u''_{yy} = 0$.

12. If $u = x^n\phi\left(\dfrac{y}{x}\right) + y^{-n}\psi\left(\dfrac{y}{x}\right)$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2\,xy\dfrac{\partial^2 u}{\partial y\,\partial x} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n^2u$.

106. Total differential. In the case of a function of a single variable, $y = f(x)$, we defined the differential of the function (§ 30) as the product of the derivative of the function and the increment of the variable, thus

$$dy = f'(x)\Delta x;$$

and then, from the special case when $y = x$, we found

$$dx = \Delta x.$$

So we had finally

$$dy = f'(x)dx.$$

In a similar manner, if in $u = f(x, y, z)$,[1] we consider all the variables excepting one as constant, we have the *partial differentials*

$$d_x u = \frac{\partial u}{\partial x} dx, \; d_y u = \frac{\partial u}{\partial y} dy, \; d_z u = \frac{\partial u}{\partial z} dz. \qquad (1)$$

Here the subscript indicates the only variable appearing in each case, the others being taken as constants.

For reasons which will be given later (§ 153), we shall define the *total differential* of u as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz. \qquad (2)$$

Thus if $\quad u = y^2 z^3 + y^3 z^2,$

$$du = \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$= (2\, yz^3 + 3\, y^2 z^2) dy + (3\, y^2 z^2 + 2\, y^3) dz.$$

Again, if $\quad u = \sqrt{x^2 + y^2 + z^2},$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial z} dz$$

$$= \frac{x\, dx + y\, dy + z\, dz}{\sqrt{x^2 + y^2 + z^2}}.$$

Even if x, y, z are not independent variables, the form (2) is still retained as the differential of u.

Possible changes in the form of the total differential of a function of several variables depend upon whether all the variables entering are independent or not, and upon the way in which the variables appear in the expression of the function. In what follows, some of the more important cases are considered in order to familiarize the student with the methods of finding the corresponding forms of the total differential.

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[1] Here also a function of three variables is selected as illustrative of the general case.

It is not intended, nor is it desirable, that the results be memorized. In each case we start with the form (2) and then replace the differentials of the dependent variables or auxiliary functions by their values in terms of the differentials of the independent variables and simplify the result by collecting the terms involving each of the differentials of the independent variables.

If z is a function of x and y, we have, as before,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz. \tag{2}$$

But now, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Putting this in (2), we have an alternative value for du,

$$du = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)dy, \tag{3}$$

in which the differentials of the independent variables only appear.

If y and z are functions of x, so that x is the only independent variable, we have, from (2),

$$du = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}\right)dx, \tag{4}$$

since $dy = \frac{dy}{dx}\,dx$ and $dz = \frac{dz}{dx}\,dx$. Here the letter "d" is used in the derivatives of y and z, because y and z are functions of a single variable.

If x, y, z are functions of a single variable or parameter t, we have from (2)

$$du = \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}\right)dt. \tag{5}$$

Let $u = f(v, w)$,[1] where v and w are functions of x, y, z. Then, considering u and v as the variables,

$$du = \frac{\partial u}{\partial v} dv + \frac{\partial u}{\partial w} dw.$$

But

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz,$$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz.$$

$$\therefore\ du = \left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}\right) dx + \left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}\right) dy + \left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z}\right) dz. \quad (6)$$

If in addition x, y, z are not independent variables, the form of (6) may be modified. Thus if z is a function of x and y, $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$. Incorporating this in (6) and rearranging the terms,

$$du = \left[\frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial u}{\partial w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}\right)\right] dx + \left[\frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial u}{\partial w}\left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}\right)\right] dy. \quad (7)$$

Ex. 1. $u = \log\sqrt{x^2 + y^2 + z^2}$, where $z = x^2 - y^2$. Find du.

Since $\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2 + z^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$, $\frac{\partial u}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$,

$$du = \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2}.$$

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[1] Here a function of two auxiliary functions is selected as illustrative of the general case. What is given here can be applied to functions of a single auxiliary function or of any number of them.

But $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 2\,x\,dx - 2\,y\,dy.$$

$$\therefore\ du = \frac{(x + 2\,xz)\,dx + (y - 2\,yz)\,dy}{x^2 + y^2 + z^2}.$$

Here z may be replaced by its value in terms of x and y; but frequently this is not desirable.

Ex. 2. $u = f(x^2 + 2\,xz - 2\,yz)$, where $z = \sqrt{x^2 + y^2}$. Find du.

Let $v = x^2 + 2\,xz - 2\,yz$. Then $u = f(v)$, and

$$du = f'(v)dv = f'(v)\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right)$$
$$= f'(v)[2(x + z)dx - 2\,zdy + 2(x - y)dz].$$

Here $f'(v) = \frac{df}{dv}$, the ordinary derivative of $f(v)$ with respect to v, since $f(v)$ is a function of the single variable v.

Moreover $$dz = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \frac{x\,dx + y\,dy}{z}.$$

Putting this in the expression for du, we have, after simple reductions,

$$du = \frac{f'(v)}{z}[2\,z(x + z) + 2\,x(x - y)]dx - \frac{f'(v)}{z}[2\,z^2 - 2\,y(x - y)]dy.$$

When desired, z may be replaced, wherever it occurs, by its value in terms of x and y.

Ex. 3. $u = f\left(\frac{y}{x},\ xyz\right)$. Find du.

Let $v = \frac{y}{x}$, $w = xyz$. Then $u = f(v, w)$, and

$$du = \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw$$
$$= \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right)$$
$$+ \frac{\partial f}{\partial w}\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\right).$$

Replacing the partial derivatives of v and w by their values in terms of x, y, and z, and rearranging the terms, we have

$$du = \left(-\frac{y}{x^2}\frac{\partial f}{\partial v} + yz\frac{\partial f}{\partial w}\right)dx + \left(\frac{1}{x}\frac{\partial f}{\partial v} + xz\frac{\partial f}{\partial w}\right)dy + xy\frac{\partial f}{\partial w}dz.$$

EXERCISES

Find du in the following cases:

1. $u = x^3 + 3y^2z - 6xz^2 + 12xyz.$
2. $u = \sin^{-1}\frac{y}{x}.$
3. $u = \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{x}{y}.$
4. $u = x^{y+z}.$
5. $u = x^y y^z z^x.$
6. $u = e^x \log yz$, where $z = \sin xy$.
7. $u = x^3 - 12yz^2 + 6xyz + 3y^2$, where $y = \sin x$, $z = \cos x$.
8. $u = f\left(x^2 + y^2, \frac{x+y}{z}\right).$
9. $u = f[\sin(x+y), \cos(y-z), \log(x+y+z)].$
10. If (x, y) are the rectangular coördinates of a point and (ρ, θ) are the polar coördinates of the same point, show that
    (a) $x\,dx + y\,dy = \rho\,d\rho$,
    (b) $x\,dy - y\,dx = \rho^2\,d\theta$,
    (c) $dx^2 + dy^2 = d\rho^2 + \rho^2\,d\theta^2$.

107. Approximate value of Δu. Relative error. As in the case of a function of a single variable, the total differential du of the function $u = f(x, y, z)$ is not the same, in general, as its actual increment Δu, corresponding to Δx, Δy, Δz, the increments of the variables (or dx, dy, dz, which are the same things when the variables are independent). The differential du is known as the *principal part* of Δu; and if all the first partial derivatives of the function are continuous, du differs from Δu by as small a quantity as one pleases when each of the increments of the variables is made sufficiently small, as we shall see in § 153. In other words, for small values of the increments of the variables, the value of du is a good approximation of the corresponding value of Δu.

Thus if x, y, z are the three dimensions of a rectangular box, its volume is $u = xyz$. If each of the dimensions is changed slightly, by the amounts dx, dy, dz respectively,

$$\Delta u = yz\,dx + zx\,dy + xy\,dz + x\,dy\,dz + y\,dz\,dx + z\,dx\,dy + dx\,dy\,dz.$$

When dx, dy, dz are very small, powers and products of these are neglible, in a first approximation, in comparison with the increments themselves. Hence the differential

$$du = yz\,dx + zx\,dy + xy\,dz$$

is a good approximation of Δu, for sufficiently small values of dx, dy, dz.

To take a concrete case, let $x = 3$, $y = 4$, $z = 5$; and $dx = 0.1$, $dy = -0.05$, $dz = 0.08$. Then

$$\Delta u = 2.2046, \text{ while } du = 2.21.$$

The principal part of the increment of the function is a good first approximation in estimating the error in the function due to errors in the measured or estimated values of the variable. Moreover, we shall be interested in the relative error $\dfrac{\Delta u}{u}$ [of which $\dfrac{du}{u}$ will be a convenient approximation], corresponding to the relative errors $\dfrac{dx}{x}, \dfrac{dy}{y}, \dfrac{dz}{z}$ in the variables.

In the case of the rectangular box, this is very easy to calculate; for

$$\frac{du}{u} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}.$$

Here the relative error in the calculated value of the volume of the box, using the given dimensions, is approximately the sum of the relative errors in these dimensions.

$$\frac{\Delta u}{u} = 0.03674, \text{ while } \frac{du}{u} = 0.03683.$$

EXERCISES

1. Find the change in volume of a cylindrical bar 3 feet long and 1 inch in diameter, if it is elongated 1 inch, and at the same time the diameter increases 0.03 inch. What is the principal part of the change?

2. A cigar box is made of wood $\frac{1}{8}$ of an inch thick. If it is 9 inches long, 5 inches wide, and $2\frac{1}{2}$ inches deep, interior measurement, what is the approximate amount of material used? What, approximately, is the relative error of your answer?

3. A cylindrical coffee canister is made of tin 0.03 inch thick. If it is 6 inches high and 4 inches in diameter, what is the approximate amount of tin used? What, approximately, is the relative error of your answer?

4. The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

What, approximately, is the error in T due to small errors in l and g?

5. The range of a projectile on a horizontal plane is

$$R = \frac{v_0^2 \sin 2\alpha}{g}.$$

What, approximately, is the error in R due to small errors in v_0, α, and g?

6. The area of a triangle in terms of its sides is

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a + b + c$. What, approximately, is the error in S due to small errors in a, b, c? What is this in particular when a, b, and c are taken respectively as 5, 6, and 7, the first two being 1 % too large, and the last 1.5 % too small?

7. Find the approximate erorr and relative error in the case of a triangle calculated from the given parts $a = 5$, $b = 10$, $C = 30°$, if $da = 0.1$, $db = 0.2$, $dC = 1'$ of arc. Show that the combined error in S may be as great as 2.8 % if a, b, C are liable to errors of 1 % each.

8. In determining the volume of a cylindrical wire, do the errors of measurement of the length and the diameter of the wire contribute equally to the relative error of the volume?

108. Various types of derivatives of a function of several variables. If

$$u = f(x, y, z),$$

and x, y, and z are independent variables, the only derivatives of u of the first order to be considered are the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ already presented in § 104.

If all the variables are not independent, then other derivatives of the first order may arise. Thus, suppose that $z = \phi(x, y)$, then $u = f[x, y, \phi(x, y)]$, and its partial derivative with respect to x, that is, its derivative with respect to x considering y as a constant, is evidently not the same as $\frac{\partial u}{\partial x}$, the derivative of $u = f(x, y, z)$ with respect to x, when y and z are both constant. The same may be said of the respective partial derivatives of $u = f[x, y, \phi(x, y)]$ and $u = f(x, y, z)$ with respect to y.

In § 30 we emphasized the fact that when u is a function of a single variable, say $u = f(x)$, its derivative, $f'(x) = \frac{du}{dx}$, is the quotient of the differential of the function and that of the variable. Considering now the partial derivative $\frac{\partial u}{\partial x}$ of

$u=f(x, y, z)$, it should be emphasized just as clearly that this is not the quotient of ∂u and ∂x. As a matter of fact the symbols ∂u and ∂x have no meaning separately. But if we recall (§ 106) that the partial differential $d_x u$ is defined as

$$d_x u=\frac{\partial u}{\partial x}dx,$$

we have

$$\frac{\partial u}{\partial x}=\frac{d_x u}{dx}; \tag{1}$$

which enables us to say that *the partial derivative of u with respect to x is equal to the quotient of the partial differential of u with respect to x and the differential of x.*

An analogous statement can be made for the partial derivative of u with respect to each of the other variables.

Use may be made of this fact in finding the partial derivative with respect to a variable if the function is not given directly in terms of the variables, but by means of one or several auxiliary functions. Thus if

$$u=f(v, w), \tag{2}$$

we have from (6), § 106,

$$du=\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}\right)dx+\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}\right)dy$$
$$+\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial z}\right)dz.$$

Hence, using (1) and its analogues,

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x},\quad \frac{\partial u}{\partial y}=\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y},$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial z}. \tag{3}$$

Similarly if
$$u = f(v, w, p), \tag{4}$$

$$\left\{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial z} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial z}. \end{aligned}\right. \tag{5}$$

Of course, if an auxiliary function is independent of any of the variables, its partial derivatives with respect to each of these is zero and does not appear in (3) or (5). For example, suppose that in (4) $v = x$, $w = y$, $p = z = \phi(x, y)$.

Then corresponding to

$$u = f(x, y, z) \text{ where } z = \phi(x, y), \tag{6}$$

we have the two variables x and y only to consider, and from (5) above, or from (3), § 106,

$$\left\{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}, \quad \text{since } \frac{\partial x}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0, \\ \frac{\partial u}{\partial y} &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}, \quad \text{since } \frac{\partial x}{\partial y} = 0 \text{ and } \frac{\partial y}{\partial y} = 1. \end{aligned}\right. \tag{7}$$

Heretofore we have used $\frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial x}$ interchangeably. But in (7) this is evidently not the case. Here $\frac{\partial f}{\partial x}$ is the partial derivative of u or of f when all the variables appearing are considered independent temporarily; while $\frac{\partial u}{\partial x}$ is the partial derivative of u with respect to x when the only variables considered independent are x and y, and z is supposed to have been replaced by its value in terms of these.

This dual use of the partial derivative symbol is not uncommon. In what follows, an attempt is made to avoid this dual use by the introduction of suitable nomenclature and symbols.

We shall reserve the name "partial derivative of u with respect to x" and "$\dfrac{\partial u}{\partial x}$", the usual symbol for it, for the case when all the variables in $f(x, y, z)$, the functional representation of u, are taken as independent. If not all the variables are independent, we shall speak of *intermediate partial derivatives* which arise when an auxiliary variable (or more of them) is replaced in f by its value in terms of the remaining variables and partial derivatives of the resulting form of the function are taken with respect to one or another of these remaining variables. Thus in (7) we shall call the left-hand members *intermediate partial derivatives with respect to x and y*, respectively. Indicating by subscripts the variables which are *not* considered constant, we shall write these

$$\left(\frac{\partial u}{\partial x}\right)_{x,z}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x},\quad \left(\frac{\partial u}{\partial y}\right)_{y,z}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}. \tag{7'}$$

As another illustration of importance consider

$$u=f(x, y, z, p, q). \tag{8}$$

If z is a function of x and y, and p and q are taken as constants, we have (7′) as before. But if, in addition, we consider p and q as functions of x, y, and z, we have

$$\begin{cases}\left(\dfrac{\partial u}{\partial x}\right)_{x,z,p,q}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial x}+\dfrac{\partial f}{\partial p}\left(\dfrac{\partial p}{\partial x}+\dfrac{\partial p}{\partial z}\dfrac{\partial z}{\partial x}\right)+\dfrac{\partial f}{\partial q}\left(\dfrac{\partial q}{\partial x}+\dfrac{\partial q}{\partial z}\dfrac{\partial z}{\partial x}\right),\\ \left(\dfrac{\partial u}{\partial y}\right)_{y,z,p,q}=\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y}+\dfrac{\partial f}{\partial p}\left(\dfrac{\partial p}{\partial y}+\dfrac{\partial p}{\partial z}\dfrac{\partial z}{\partial y}\right)+\dfrac{\partial f}{\partial q}\left(\dfrac{\partial q}{\partial y}+\dfrac{\partial q}{\partial z}\dfrac{\partial z}{\partial y}\right).\end{cases} \tag{9}$$

Of course, the right-hand members of (7′) and (9) cannot be written until the nature of the variables is known. But, when this is known, the left-hand members of these equations are convenient symbols for the expressions in the corresponding right-hand members.

Finally, if all the variables appearing in f are functions of one of them, or of some other variable (or parameter), u is actually a function of a single variable, and its derivative with respect to that variable is known as its *total derivative.* Thus, if in (4) $v = x$, $w = y = y(x)$, $p = z = z(x)$, u is a function of the single variable x; and in writing its derivative as given by (5) the form "d" may be used in the expressions for the derivatives of u, y, z. Doing this, we obtain

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} \tag{10}$$

as the *total derivative* of $u = f(x, y, z)$, where $y = y(x)$, $z = z(x)$.

Or, if in (4), $v = x = x(t)$, $w = y = y(t)$, $p = z = z(t)$, we have from (5)

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \tag{11}$$

as the *total derivative* of $u = f(x, y, z)$ where x, y, and z are functions of the parameter t.

In what has been given here, the values of intermediate partial derivatives and of total derivatives have been found by means of (5). They can be obtained also from the differential of u as follows:

In § 106 attention was called to the fact that whether x, y, and z are independent variables or not, the total differential of $u = f(x, y, z)$ may be written

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz,$$

which is the sum of the partial differentials

$$d_xu = \frac{\partial u}{\partial x}dx, \quad d_yu = \frac{\partial u}{\partial y}dy, \quad d_zu = \frac{\partial u}{\partial z}dz.$$

And, as pointed out in (1) of the present section, the partial derivatives of u are thus expressible as the quotients of these partial differentials and the corresponding differentials of the variables.

In § 106 it was also shown that when the variables are not independent, other forms for du are possible; for example, formulas (3), (4), (5), (7) of that section. In such a case we shall call the aggregate of the terms involving the differential of one of the variables, in the expression for du, an *intermediate differential* of the function with respect to that variable and the others connected with it, and shall indicate it by affecting the "d" by a set of subscripts of which the first is the variable of reference and the following ones are the other variables associated with it. Thus in (3), § 106,

$$du = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)dy,$$

we have the intermediate differentials

$$d_{x,z}u = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)dx \quad \text{and} \quad d_{y,z}u = \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)dy.$$

Comparing this with (7′), it is seen readily that

$$\left(\frac{\partial u}{\partial x}\right)_{x,z} = \frac{d_{x,z}u}{dx}, \quad \left(\frac{\partial u}{\partial y}\right)_{y,z} = \frac{d_{y,z}u}{dy}.$$

In general it may be stated that whenever the differential of a function is expressed as a sum of intermediate differentials, the quotient of any of the intermediate differentials and the differential of the corresponding variable is equal to the corresponding intermediate partial derivative of the function with respect to that variable. Thus from (7), § 106, we see at once that for $u = f(v, w)$, where v and w are functions of x, y, and z, and z is a function of x and y,

$$\left(\frac{\partial u}{\partial x}\right)_{x,z} = \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial u}{\partial w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}\right),$$

$$\left(\frac{\partial u}{\partial y}\right)_{y,z} = \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial u}{\partial w}\left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}\right).$$

If all the variables involved are functions of one of them, or of some other variable (or parameter), it is obvious that the total derivative of the function with respect to the single independent variable is the quotient of its total differential and the differential of that variable. Thus compare formulas (4) and (5) of § 106 and (10) and (11) of the present section.

To find the second and higher partial derivatives of a function of several variables or of a function of one or more functions of those variables, the principles of this section may be employed, remembering that a second derivative is a first derivative of a first derivative, and so on.

As an illustration of the use of formulas (5) let it be required to find the effect of the transformation of variables $x = \rho \cos \theta$, $y = \rho \sin \theta$ upon $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, where u is a given function of x and y, and after transformation becomes a function of ρ and θ.

We shall have occasion to use the transformation of variables in the form $\rho = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$.

Using (5), we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial \rho}\frac{x}{\sqrt{x^2 + y^2}} - \frac{\partial u}{\partial \theta}\frac{y}{x^2 + y^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial \rho}\frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial u}{\partial \theta}\frac{x}{x^2 + y^2}.$$

Squaring and adding, we have the important formula

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Differentiating again

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial^2 u}{\partial \rho^2}\frac{\partial \rho}{\partial x} + \frac{\partial^2 u}{\partial \theta\,\partial \rho}\frac{\partial \theta}{\partial x}\right)\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \rho}\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} \\ &\quad - \left(\frac{\partial^2 u}{\partial \theta\,\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{y}{x^2+y^2} - \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x}\frac{y}{x^2+y^2} \\ &= \frac{\partial^2 u}{\partial \rho^2}\frac{x^2}{x^2+y^2} - 2\frac{\partial^2 u}{\partial \theta\,\partial \rho}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\partial^2 u}{\partial \theta^2}\frac{y^2}{(x^2+y^2)^2} \\ &\quad + \frac{\partial u}{\partial \rho}\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\partial u}{\partial \theta}\frac{2\,xy}{(x^2+y^2)^2}.\end{aligned}$$

It will be left as an exercise to show that

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \rho^2}\frac{y^2}{x^2+y^2} + 2\frac{\partial^2 u}{\partial \theta\,\partial \rho}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\partial^2 u}{\partial \theta^2}\frac{x^2}{(x^2+x^2)^2} \\ &\quad + \frac{\partial u}{\partial \rho}\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} - \frac{\partial u}{\partial \theta}\frac{2\,xy}{(x^2+y^2)^2}.\end{aligned}$$

Adding these and making obvious reductions, we arrive at the important formula

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}.$$

EXERCISES

1. If $u = x^2 - y^2$ and $x = r\cos t$, $y = r\sin t$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial t}$.

2. If $u = e^{xy}$ and $x = \sqrt{r^2+s^2}$, $y = \tan^{-1}\dfrac{s}{r}$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$.

3. If $x = x'\cos\alpha - y'\sin\alpha$ and $y = x'\sin\alpha + y'\cos\alpha$, show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2.$$

4. Show that the differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\,xy^2\frac{\partial u}{\partial x} + 2(y - y^3)\frac{\partial u}{\partial y} + x^2y^2u = 0$$

is left unaltered in form by the transformation of variables $r = xy,\ s = \frac{1}{y}\cdot$

5. Show that the change of variables $x = e^\rho \cos\theta,\ y = e^\rho \sin\theta$ transforms $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ into $e^{-2\rho}\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2}\right)\cdot$

109. Derivatives of implicit functions. If y is defined as a function of x by the relation

$$f(x, y) = 0, \tag{1}$$

we have, on taking the derivative with respect to x,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

Hence

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \tag{2}$$

This is an important formula. Similar formulas for the partial derivatives of implicit functions of several variables may be found. Thus if z is defined as a function of x and y by the relation

$$f(x, y, z) = 0, \tag{3}$$

we have, on taking derivatives with respect to x and y respectively,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0.$$

Hence

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}. \tag{4}$$

Ex. 1. Given $(x^2+y^2)^2 - a^2(x^2-y^2) = 0$, find $\frac{dy}{dx}$. Writing the given equation as $f(x, y) = 0$, we have

$$\frac{\partial f}{\partial x} = 4\,x(x^2+y^2) - 2\,a^2x, \qquad \frac{\partial f}{\partial y} = 4\,y(x^2+y^2) + 2\,a^2y.$$

Hence, from (2), $$\frac{dy}{dx} = -\frac{x[2\,(x^2+y^2) - a^2]}{y[2(x^2+y^2) + a^2]}.$$

Ex. 2. Given $ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Writing the given equation $F(x, y, z) = 0$, we have

$$\frac{\partial F}{\partial x} = 2(ax + hy + gz), \qquad \frac{\partial F}{\partial y} = 2(hx + by + fz),$$

$$\frac{\partial F}{\partial z} = 2\,(gx + fy + cz).$$

$$\therefore\ \frac{\partial z}{\partial x} = -\frac{ax + by + gz}{gx + fy + cz}, \qquad \frac{\partial z}{\partial y} = \frac{hx + by + fz}{gh + fy + cz}.$$

EXERCISES

1. Obtain $\frac{dy}{dx}$ from the following relations:

(*a*) $ax^2 + 2\,bxy + cy^2 = 1$.

(*b*) $e^x + e^y = 2\,xy$.

(*c*) $x^3 + y^3 - 3\,axy = 0$.

(*d*) $\sqrt{x^2+y^2} - a\tan^{-1}\frac{y}{x} = 0$.

2. Obtain $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ from the following relations:

(*a*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(*b*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(*c*) $xy + yz + zx = 1$.

(*d*) $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$.

(*e*) $e^x + e^y + e^z = 3\,xyz$.

3. Given $f(x, y, z) = 0$, show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.

4. Given $f(x, y, z, t) = 0$, show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = 1$.

110. Geometrical application. Let the equation of a surface be

$$f(x, y, z) = 0. \tag{1}$$

Taking the differential of both members, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \tag{2}$$

If P, a point on the surface, moves along a curve PQ in the surface, the direction cosines of the tangent PT are proportional to (see Section II of the Appendix)

$$dx : dy : dz. \tag{3}$$

The direction cosines of the tangent line of any curve on the surface at P are thus connected by the relation (2). But for the values of the coördinates at P, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ assume definite values. These may be considered proportional to the direction cosines of a definite line PN through P. The relation (2) shows that the lines PT and PN are mutually perpendicular. We thus see that the tangent lines to all curves on the surface at P are perpendicular to the same line PN. These tangent lines therefore lie in a plane, which is known as the *tangent plane* to the surface at P, and their common perpendicular PN is known as the *normal* to the surface at P. We have thus arrived at the following

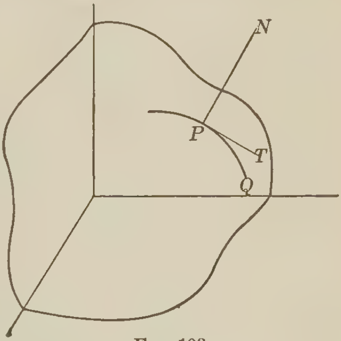

FIG. 103

geometrical significance of the partial derivatives of a function of three variables:

If $P \equiv (x, y, z)$ *is a point on the surface* $f(x, y, z) = 0$, *the direction cosines of the normal to the surface at* P *are proportional to* $\left(\frac{\partial f}{\partial x}\right)_1 : \left(\frac{\partial f}{\partial y}\right)_1 : \left(\frac{\partial f}{\partial z}\right)_1$, where $\left(\frac{\partial f}{\partial x}\right)_1$ is the value of $\frac{\partial f}{\partial x}$ when $x = x_1$, $y = y_1$, $z = z_1$; and similarily for the other two. The equations of the normal at P are then

$$\frac{x - x_1}{\left(\frac{\partial f}{\partial x}\right)_1} = \frac{y - y_1}{\left(\frac{\partial f}{\partial y}\right)_1} = \frac{z - z_1}{\left(\frac{\partial f}{\partial z}\right)_1}; \tag{4}$$

and the equation of the tangent plane at P is

$$\left(\frac{\partial f}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial f}{\partial y}\right)_1 (y - y_1) + \left(\frac{\partial f}{\partial z}\right)_1 (z - z_1) = 0. \tag{5}$$

Thus to find the equations of the normal line and the tangent plane to $2x^2 - 3y^2 + 2yz = 3z - 1$ at $(3, -2, 1)$, we write

$$f(x, y, z) = 2x^2 - 3y^2 + 2yz - 3z + 1 = 0.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_1 = (4x)_1 = 12, \quad \left(\frac{\partial f}{\partial y}\right)_1 = (-6y + 2z)_1 = 14,$$

$$\left(\frac{\partial f}{\partial z}\right)_1 = (2y - 3)_1 = -7.$$

Hence the equations of the normal are

$$\frac{x - 3}{12} = \frac{y + 2}{14} = \frac{z - 1}{-7};$$

and the equation of the tangent plane is

$$12(x - 3) + 14(y + 2) - 7(z - 1) = 0$$

or

$$12x + 14y - 7z = 1.$$

At the point $(0, 1, -2)$ on the surface the direction cosines of the normal are proportional to $0 : -10 : -1$, or to $0 : 10 : 1$.

Hence the equations of the normal in the symmetrical form are

$$\frac{x}{0}=\frac{y-1}{10}=\frac{z+2}{1}.$$

The normal is thus the line in the plane $x=0$ cut out by the plane $\frac{y-1}{10}=z+2$ or $y-10z=21$.

The equation of the tangent plane is

$$10(y-1)+z+2=0 \text{ or } 10y+z-8=0.$$

EXERCISES

Find the equations of the normal line and tangent plane to each of the following surfaces at the points given:

1. Sphere $(x+1)^2+y^2+(z-3)^2=36$, at $(3, 4, 5)$.

2. Ellipsoid $2x^2+3y^2+z^2-4x+6y-8z+20=0$, at $(1, -1, 3)$.

3. Hyperboloid $3x^2-2y^2+z^2-5x+z=0$ at $(0, 0, 0)$; also at $(2, 1, -1)$.

4. Elliptic paraboloid $3x^2+z^2=4y$ at $(0, 0, 0)$; also at $(1, 3, -3)$.

5. Cylinder $x^2-3xy+2y^2-2x+3y-1=0$ at $(1, -1, 1)$.

6. Cone $x^2-y^2+2z^2+2xy+3xz+yz=0$ at $(1, 1, -1)$.

7. The cubic surface $xyz=1$ at $(1, -1, -1)$; also at $(\frac{1}{2}, 2, 1)$.

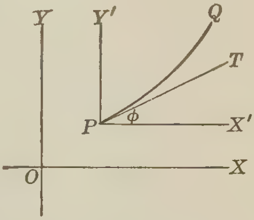

FIG. 104

111. Directional derivative. Let

$$u=f(x, y) \qquad (1)$$

be a single-valued function of x and y. The relation (1) assigns to u a definite value at each point P of the plane. As P moves along a curve PQ, the value of u will change, as a rule. The rate of change of u with respect to the arc s of the curve PQ, that is, the derivative $\frac{du}{ds}$, is known as the *directional derivative* of u. The analytic expression for it is readily obtained by assuming the equation of the curve PQ given in the parametric form

$$x = x(s),\ y = y(s).$$

The function u then takes the form

$$u = f[x(s),\ y(s)],$$

and

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} \tag{2}$$

is the directional derivative.

The directional derivative of a function at a point is evidently the same whether taken along a curve through the point or along the tangent to the curve at that point. It is *a function of the direction* at the point.

In particular, if the direction is that of the x-axis, $x = s$ and $y =$ constant; whence

$$\frac{du}{ds} = \frac{\partial u}{\partial x}.$$

Similarly along the direction of the y-axis

$$\frac{du}{ds} = \frac{\partial u}{\partial y}.$$

The partial derivatives of u are thus seen to be the directional derivatives of u in the directions of the respective axes.

If ϕ is the inclination of the tangent PT to the curve at P, $\frac{dx}{ds} = \cos\phi$ and $\frac{dy}{ds} = \sin\phi$. Hence (2) may be written

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\cos\phi + \frac{\partial u}{\partial y}\sin\phi, \tag{3}$$

which says that *the directional derivative along a line is equal to the sum of the projections upon this line of the directional derivatives in the direction of the coördinate axes.*

Keeping the point P fixed, the directional derivative is a function of ϕ only. Writing it

$$\frac{du}{ds} = F(\phi) = \frac{\partial u}{\partial x}\cos\phi + \frac{\partial u}{\partial y}\sin\phi, \tag{4}$$

several facts in connection with it may be noted.

I. *The directional derivative of* $u = f(x, y)$ *is zero along the curve* $f(x, y) =$ *constant.*

This follows at once from the definition of the directional derivative. It may also be obtained from (4), as follows:

$F(\phi) = 0$ if $\tan\phi = -\dfrac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$. But this is the slope of the tangent of $f(x, y) =$ constant at a point on the curve [(2), § 109].

II. *The directional derivative of* $u = f(x, y)$ *is a maximum along the normal to the curve* $f(x, y) =$ *constant.*

In order that $F(\phi)$ be a maximum or minimum

$$F'(\phi) = -\frac{\partial u}{\partial x}\sin\phi + \frac{\partial u}{\partial y}\cos\phi = 0 \tag{5}$$

or

$$\tan\phi = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}. \tag{6}$$

This direction is evidently perpendicular to that of the tangent to the curve.

Differentiating again,

$$F''(\phi) = -\frac{\partial u}{\partial x}\cos\phi - \frac{\partial u}{\partial y}\sin\phi = -F(\phi). \tag{7}$$

From (4) it is obvious that the directional derivative of u changes sign when the direction along the curve is reversed, for

$F(\phi + \pi) = - F(\phi)$. Thus going along the normal in one direction, the directional derivative is positive, in the opposite direction it is negative. Evidently, from (7), in the former direction $F(\phi)$ is a maximum, in the latter direction it is an algebraic minimum. But numerically the two are the same. The statement II is intended to refer to the absolute value of the derivative.

It is customary to speak of the derivative in that direction along the normal in which it is positive as the *normal derivative*[1] of the function and to designate it by $\frac{du}{dn}$. From (6) we see that *along the normal*

$$\sin\phi = \frac{\frac{\partial u}{\partial y}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}}, \quad \cos\phi = \frac{\frac{\partial u}{\partial x}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}}.$$

Putting these in (3), the normal derivative is seen to be

$$\frac{du}{dn} = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}. \tag{8}$$

III. *The directional derivative along any direction is equal to the normal derivative multiplied by the cosine of the angle between that direction and the normal.*

In Fig. 105, $u = c$ is the curve along which u has the value it assumes at $P \equiv (x, y)$. As this point moves to $Q \equiv (x + \Delta x, y + \Delta y)$ in the direction PS, the corresponding change in s is $\Delta s = PQ$ and that in n is $\Delta n = PR$, where RQ is the curve $u = c + \Delta c$ passing through Q. It can be proved that

$$\underset{\Delta s \to 0}{L} \frac{\Delta n}{\Delta s} = \frac{dn}{ds} = \cos\psi.$$

---

[1] The term "normal derivative of a function u with respect to a curve" is often used for the directional derivative of the function along the normal to the curve. We are using the term "normal derivative of u" in the special sense where the curve is $u =$ constant.

We found that

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}. \tag{2}$$

If, instead of having x and y expressed directly in terms of s, we have them given in terms of n which itself is a function of s, we have

$$\begin{aligned}\frac{du}{ds}&=\frac{\partial u}{\partial x}\frac{dx}{dn}\frac{dn}{ds}+\frac{\partial u}{\partial y}\frac{dy}{dn}\frac{dn}{ds}\\&=\left(\frac{\partial u}{\partial x}\frac{dx}{dn}+\frac{\partial u}{\partial y}\frac{dy}{dn}\right)\cos\psi\\&=\frac{du}{dn}\cos\psi.\end{aligned}$$

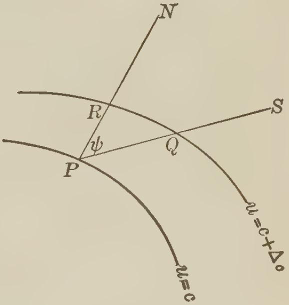

FIG. 105

All that has been given here can be extended to the function

$$u=f(x, y, z),$$

where x, y, z are taken as the coördinates of a point in space. But we shall simply mention here the following obvious facts:

The directional derivative of u along the curve $x=x(s)$, $y=y(s)$, $z=z(s)$ is

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial y}\frac{dy}{ds}+\frac{\partial u}{\partial z}\frac{dz}{ds}. \tag{2'}$$

Or, noting that $\frac{dx}{ds}=\cos\alpha$, $\frac{dy}{ds}=\cos\beta$, $\frac{dz}{ds}=\cos\gamma$, the direction cosines of the curve at the point, we may write

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma. \tag{3'}$$

Evidently $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ are the directional derivatives of u in the directions of the respective coördinate axes.

EXERCISES

1. Find the directional derivative of $u = x^2 - y^2$ in the direction $\phi = 45°$ at $(3, -1)$; also at $(3, 3)$, and at $(3, -3)$. Also find the normal derivatives of the function at these points.

2. Find the directional derivative of $u = x^2y$ in the direction $\phi = 150°$ at $(1, 2)$; also at $(2, -\sqrt{3})$, and at $(0, -2)$. Also find the normal derivatives of the function at these points.

3. Show that the sum of the squares of the directional derivatives of u along any two mutually perpendicular directions is equal to the square of the normal derivative of u.

4. Show that the normal derivative of $u = f(x, y, z)$ is equal to

$$\frac{du}{dn} = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}.$$

112. Maxima and minima of functions of several variables. Analogous to the case of a function of a single variable, a function of several variables is said to have a *maximum* for a set of values of the variables if its value corresponding to them is greater than the values it assumes for *all neighboring* values of the variables. Thus if $u = +\sqrt{1 - x^2 - y^2}$, it is evident that, corresponding to $x = 0$, $y = 0$, $u = 1$ is larger than any value u assumes for other values of x and y near zero.

Again if $u = (x^2 + y^2)(x^2 + y^2 - 1)$, it is clear that, corresponding to $x = 0$, $y = 0$, $u = 0$ is larger than any values u assumes for values of x and y near zero. To be sure, for values of x and y making $x^2 + y^2 - 1$ positive, u assumes positive and therefore larger values. But according to the definition of a maximum, only neighboring values of the variables are to be taken into account.

Analytically we may write: The function $u = f(x, y, z)$ will have a maximum at (x_1, y_1, z_1) if $f(x_1 + \Delta x,\ y_1 + \Delta y, z_1 + \Delta z)$

$> f(x_1, y_1, z_1)$ for all sufficiently small values of Δx, Δy, Δz. How small these must be restricted to be depends upon the nature of $f(x, y, z)$.

In an analogous manner, $u = f(x, y, z)$ will have a *minimum* at (x_1, y_1, z_1) if $f(x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z) < f(x_1, y_1, z_1)$ for all sufficiently small values of Δx, Δy, Δz.

In particular, if $f(x_1, y_1, z_1)$ is to be a maximum or a minimum, it must be such for each of the functions $f(x, y_1, z_1)$, $f(x_1, y, z_1)$, $f(x_1, y_1, z)$, in which one variable alone appears. The necessary conditions for these are

$$\left(\frac{df\ (x, y_1, z_1)}{dx}\right)_{x=x_1} = 0,\ \left(\frac{df\ (x_1, y, z_1)}{dy}\right)_{y=y_1} = 0,$$

$$\left(\frac{df\ (x_1, y_1, z)}{dz}\right)_{z=z_1} = 0;$$

or, in the notation introduced in § 110, a set of necessary conditions for a maximum or a minimum at (x_1, y_1, z_1) is

$$\left(\frac{\partial f}{\partial x}\right)_1 = 0,\ \left(\frac{\partial f}{\partial y}\right)_1 = 0,\ \left(\frac{\partial f}{\partial z}\right)_1 = 0. \tag{1}$$

Since the total differential of u is

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

it follows that, whenever the conditions (1) hold, du vanishes for $x = x_1$, $y = y_1$, $z = z_1$. Hence *the vanishing of the total differential of a function for a set of values of the variables is a necessary condition that the function should have a maximum or minimum for those values of the variables.*

This is a concise way of expressing conditions (1). Besides it should be noted that, while in (1) the variables x, y, z are supposed to be independent, in $du = 0$ all the variables appearing need not be.

These conditions are not always sufficient, just as the vanishing of the derivative of a function of a single variable is not always sufficient for the existence of a maximum or a minimum. The general discussion for a function of several variables is more complicated than that for a function of one variable and will not be given here. In the cases considered here, it will be possible to tell from the nature of the problem whether the conditions (1) lead to a set of values of the variables corresponding to a maximum or minimum or not. The following exercises will illustrate.

Ex. 1. Find the relative dimensions of an open rectangular box of maximum content requiring a given amount of material. If the dimensions of the base are x and y, and the depth is z, the content is

$$V = xyz. \tag{2}$$

Neglecting the thickness of the material, the quantity required is

$$A = xy + 2\,xz + 2\,yz. \tag{3}$$

This determines one of the variables as a function of the other two; so we have two independent variables. From (3)

$$z = \frac{A - xy}{2(x+y)}. \tag{4}$$

Putting this in (2), we have

$$V = \frac{Axy - x^2y^2}{2(x+y)}.$$

For a maximum content we must have

$$\frac{\partial V}{\partial x} = \frac{Ay^2 - x^2y^2 - 2\,xy^3}{2(x+y)^2} = 0; \quad \frac{\partial V}{\partial y} = \frac{Ax^2 - x^2y^2 - 2\,x^3y}{2(x+y)^2} = 0.$$

Since $x = 0$ and $y = 0$ are obviously not allowable values, these relations reduce to

$$x^2 + 2\,xy = A \text{ and } y^2 + 2\,xy = A. \tag{5}$$

Hence, $x^2 = y^2$ is a required relation. But from the nature of the problem, the dimensions must be positive.

$$\therefore\ x = y = \sqrt{\frac{A}{3}}.$$

Putting these values in (4), we have

$$z=\frac{1}{2}\sqrt{\frac{A}{3}}=\frac{x}{2}=\frac{y}{2}.$$

Another method may be mentioned here, because it is often simpler in application.

Using the fact that dV must vanish for a maximum, we have

$$dV=yz\,dx+xz\,dy+xy\,dz=0.$$

Moreover, from (3) we have, on differentiating,

$$(y+2z)\,dx+(x+2z)\,dy+2(x+y)\,dz=0.$$

From these it follows that

$$\frac{y+2z}{yz}=\frac{x+2z}{xz}=\frac{2(x+y)}{xy},$$

or

$$\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}=\frac{2}{y}+\frac{2}{x},$$

$$\therefore\ x=y=2z.$$

Ex. 2. Show that the line along which a maximum or a minimum distance from a point (a, b, c) not on a surface $f(x, y, z)=0$ to the surface is measured is normal to the surface.

If (x, y, z) is the point where the line meets the surface, its distance from (a, b, c) is

$$l=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

There are two independent variables only because of the relation $f(x, y, z)=0$. Instead of l, consider the function

$$l^2=(x-a)^2+(y-b)^2+(z-c)^2.$$

Using the second method of Ex. 1, we have

$$\tfrac{1}{2}\,d(l^2)=(x-a)\,dx+(y-b)\,dy+(z-c)\,dz=0,$$

and

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0;$$

whence

$$\frac{\dfrac{\partial f}{\partial x}}{x-a}=\frac{\dfrac{\partial f}{\partial y}}{y-b}=\frac{\dfrac{\partial f}{\partial z}}{z-c}.$$

This is the desired result.

Ex. 3. Find the maximum value of $u = 3\,axy - x^3 - y^3$.

Here $\dfrac{\partial u}{\partial x} = 3\,ay - 3\,x^2 = 0$, $\dfrac{\partial u}{\partial y} = 3\,ax - 3\,y^2 = 0$. Solving these, we have

$$x = y = 0, \text{ or } x = y = a.$$

A little consideration assures one that $x = 0$, $y = 0$ corresponds to neither a maximum nor a minimum:

When $x = y = 0$, $u = 0$.

When $x = \Delta x$ and $y = \Delta y$, $\Delta u = 3\,a\,\Delta x\,\Delta y - \Delta x^3 - \Delta y^3$. For Δx and Δy sufficiently small, the term of lowest degree $3\,a\,\Delta x\,\Delta y$ will determine the algebraic sign of Δu. Since this is positive when Δx and Δy have the same sign, and negative when they have opposite signs, it follows that $u = 0$ corresponding to $x = y = 0$ is neither a maximum nor a minimum.

On the other hand, $x = y = a$ determines a maximum value of the function. For when

$$x = a + \Delta x \text{ and } y = a + \Delta y,$$

$$\Delta u = -3\,a(\Delta x^2 - \Delta y^2 - \Delta x\,\Delta y) - \Delta x^3 - \Delta y^3.$$

Here the sign of $-3\,a(\Delta x^2 + \Delta y^2 - \Delta x\,\Delta y)$ determines the sign of Δu. But the sum of the squares of two real numbers is always greater than the product of those numbers.[1] Hence Δu is negative for all small values of Δx and Δy, and $u = a^3$, corresponding to $x = y = a$, is a maximum.

---

[1] As a matter of fact it can be proved that the sum of the squares of two real numbers is greater than twice the product of the numbers if they are unequal. Thus, if $a \neq b$, let $a > b$; then $a - b > 0$ and $a^2 - 2\,ab + b^2 > 0$ or $a^2 + b^2 > 2\,ab$. [If $a = b$, then $a - b = 0$, and $a^2 - 2\,ab + b^2 = 0$ or $a^2 + b^2 = 2\,ab$.]

EXERCISES

1. Find the maximum value of

$$\frac{(ax+by+c)^2}{x^2+y^2+1}.$$

2. Find the minimum value of

$$x^2+xy+y^2-ax-by.$$

3. Show that the sum of the squares of the distances of a point $P \equiv (x, y)$, from three points (x_1, y_1), (x_2, y_2), (x_3, y_3) is a minimum when P is the center of gravity of the triangle having those points for vertices.

4. Divide the number k into three parts so that the product of the parts is a maximum.

5. Divide the number k into three parts x, y, z such that $x^a y^b z^c$ is a maximum.

6. Find the values of x, y, z for which

$$u = \frac{xyz}{ax+by+cz}$$

is a maximum, if x, y, z are connected by the relation $xyz = k$, a constant. Find the maximum value of u.

7. Prove that the rectangular box of given content requiring a minimum amount of material is a cube.

8. Find the shortest and the longest distance from the point $(1, 2, -1)$ to the sphere $x^2+y^2+z^2=24$.

9. Find the point in the plane $2x-3y+z+11=0$ nearest to the point $(2, 0, -1)$.

10. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

11. What is the minimum value of the sum of the squares of the distances of a variable point from the sides of a given triangle? [Let the lengths of the sides of the triangle be a, b, c, and the distances of the point from these x, y, z respectively.]

12. What is the minimum value of the sum of the squares of the distances of a variable point from the sides of a given tetrahedron? [Let the areas of the faces of the tetrahedron be A, B, C, D, and the distances of the point from these x, y, z, t respectively.]

13. Find the minimum value of $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2$ subject to the condition $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = k$.

14. Show that the triangle of maximum area inscribed in a circle is equilateral.

15. Show that the polygon of n sides inscribed in a circle must be regular if its area is to be a maximum.

16. Show that of all polygons of a given number of sides circumscribed about a circle, the regular polygon has the minimum area.

17. Show that the directional derivative of $u = f(x, y, z)$ is a maximum along the normal to the surface $f(x, y, z) = c$.

In (3′), § 111, let the variables $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be represented by single letters, such as l, m, n, respectively.

113. Homogeneous function. Euler's Theorem. A *polynomial in several variables* is the algebraic sum of terms involving positive integral powers of the variables. Thus, for example,

$$x^3y, \quad x^2 + 3xy - y^2 + x - 3, \quad ax + \sqrt{b}\,y + \frac{z}{c},$$

$$3x^2 - \tfrac{1}{5}xy + \sqrt{2}\,yz + z - 3$$

are polynomials in the variables appearing in each case. The character of the coefficients plays no rôle.

The *degree of a term* of a polynomial in several variables is the sum of the exponents of the variables involved.

If all the terms of a polynomial are of the same degree n, the polynomial is said to be *homogeneous of the degree n*. Thus

$$x^3 + y^3 - 6x^2y \text{ and } x^2 + y^2 - z^2 + 2xy - 4yz$$

are homogeneous of degree 3 and 2 respectively.

Evidently, if $f(x, y, z)$ is a homogeneous polynomial of degree n

$$f(tx, ty, tz) = t^n f(x, y, z), \tag{1}$$

where t is any multiplier whatever. This follows at once from the definition of a homogeneous polynomial. For example,

if $f(x, y) = x^3 + y^3 - 6\,x^2y$, $f(tx, ty) = t^3x^3 + t^3y^3 - 6\,t^2x^2ty$
$= t^3 f(x, y)$,

if $f(x, y, z) = x^2 + y^2 - z^2 + 2\,xy - 4\,yz$,

$f(tx, ty, tz) = t^2x^2 + t^2y^2 - t^2z^2 + 2\,tx\,ty - 4\,ty\,tz = t^2 f(x, y, z)$.

This property of homogeneous polynomials is an important one. But there is a more extended class of functions which have it.

Any function in two or more variables satisfying the relation (or identity for all multipliers t)

$$f(tx, ty, tz) = t^n f(x, y, z) \tag{1}$$

is said to be a *homogeneous function of degree n*.

The degree n need not be an integer nor need it be positive. The following are examples of homogeneous functions which are not polynomials:

$\sqrt{x^2 + y^2 + z^2}$ is homogeneous of degree 1, for

$$\sqrt{t^2x^2 + t^2y^2 + t^2z^2} = t\sqrt{x^2 + y^2 + z^2}.$$

$\sqrt{\dfrac{x + y + z}{x^2 + z^2}}$ is homogeneous of degree $-\frac{1}{2}$, for

$$\sqrt{\frac{tx + ty + tz}{t^2x^2 + t^2y^2}} = \frac{1}{\sqrt{t}}\sqrt{\frac{x + y + z}{x^2 + z^2}}.$$

$\tan^{-1}\dfrac{y}{x}$ is homogeneous of degree 0, for

$$\tan^{-1}\frac{ty}{tx} = \tan^{-1}\frac{y}{x} = t^0 \tan^{-1}\frac{y}{x}.$$

In the same way the functions $f\left(\frac{y}{x}\right)$ and $f\left(\frac{y}{x}, \frac{z}{x}\right)$ are seen to be homogeneous of degree 0.

The relation (1) is an identity, holding no matter what be the form of the multiplier t. In particular, if in (1) $t = \frac{1}{x}$,

we have
$$f\left(1, \frac{y}{x}, \frac{z}{x}\right) = \frac{1}{x^n} f(x, y, z)$$
or
$$f(x, y, z) = x^n f\left(1, \frac{y}{x}, \frac{z}{x}\right). \tag{2}$$

Similar forms can be obtained for y and z by letting $t = \frac{1}{y}$ and $t = \frac{1}{z}$.

Property (2), which is obvious in the case of homogeneous polynomials, is an important one for certain purposes for any homogeneous function. It says that *a homogeneous function of degree n in a set of variables may be expressed as the product of the nth power of one of the variables and a homogeneous function of zero degree.* Moreover the form of this second factor is readily given by (2).

Noting that (1) holds for all values of t, we may differentiate both members with respect to it. Doing this, we have[1]

$$x \frac{\partial f(tx, ty, tz)}{\partial(tx)} + y \frac{\partial f(tx, ty, tz)}{\partial(ty)} + z \frac{\partial f(tx, ty, tz)}{\partial(tx)}$$
$$= nt^{n-1} f(x, y, z).$$

In particular, when $t = 1$, this leads to the important relation

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf, \tag{3}$$

which is known as *Euler's theorem for a homogeneous function.*

---

[1] Here use is made of the general formula, analogous to (11), § 108,

$$\frac{du}{dt} = \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt},$$

where $v = tx$, $w = ty$, $p = tz$, and x, y, z are considered constant.

EXERCISES

Verify that the following functions are homogeneous, and that Euler's theorem holds for them. What is the degree of each?

1. $\dfrac{1}{x^2-y^2}$.
2. $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$.
3. $\dfrac{x+y}{\sqrt{x^2-y^2}}$.
4. $\sqrt{x}\sin\dfrac{y}{x}$.
5. $f\left(\dfrac{x+y}{z}\right)$.
6. $\log\left(\dfrac{y}{z}+\dfrac{z}{x}+\dfrac{x}{y}\right)$.
7. $\dfrac{y^2}{x}+(\log x-\log y)\sqrt{x^2+y^2}$.
8. $(x^2+y^2-z^2)\sqrt{xyz}$.
9. $e^{\frac{y}{x}}(x-y)$.
10. $x\cos\dfrac{y}{z}-ye^{\frac{x}{z}}$.

114. Exact differential. Starting with a function of two variables

$$u=f(x, y),$$

we find its differential to be

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy. \tag{1}$$

This is of the form

$$M(x, y)dx+N(x, y)dy. \tag{2}$$

Conversely, starting with a differential expression like (2), it is often of importance to know whether it is the differential of some function u of x and y. When it is, it is said to be an *exact differential.*

If u is a function of which $M\,dx+N\,dy$ is the differential, we can write

$$du=M\,dx+N\,dy. \tag{3}$$

But we also have

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy. \tag{1}$$

Since dx and dy are arbitrary increments in (1) and (3), in order that both of these relations can hold, we must have

$$M=\frac{\partial u}{\partial x} \text{ and } N=\frac{\partial u}{\partial y}. \tag{4}$$

Differentiating, we have

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x} \text{ and } \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}. \tag{5}$$

Since $\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}$, it follows that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \tag{6}$$

is a necessary condition that $M\,dx + N\,dy$ *be an exact differential.* For it resulted from the assumption that a function u exists such that equation (3) holds.

Conversely, it can be shown[1] that *the condition* (6) *is also sufficient that* $M\,dx + N\,dy$ *be an exact differential.*

Correspondingly, *the necessary and sufficient conditions that*

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dy$$

be an exact differential, that is, that it be the differential of some function $u(x, y, z)$, *are*

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}.$$

Ex. 1. Show that $(3x^2-3y)dx-(3x+4y-1)dy$ is an exact differential.

Here $\frac{\partial}{\partial y}(3x^2-3y)=-3,\quad \frac{\partial}{\partial x}(-3x-4y+1)=-3.$

Hence the expression is an exact differential.

Frequently, by a proper arrangement of the terms, one can find the function of which the given differential expression is the differential. Thus, writing it in the form

$$du = 3x^2\,dx - 3(y\,dx + x\,dy)-(4y-1)\,dy,$$

we have

$$u = x^3 - 3xy - 2y^2 + y + c.$$

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[1] A proof of this is given in the author's Elementary Treatise on Differential Equations, § 7.

Ex. 2. Show that $(x + y^2 + z^2 + 1)dx + 2\,xy\,dy + 2\,xz\,dz$ is an exact differential.

Here

$$\frac{\partial(x + y^2 + z^2 + 1)}{\partial y} = 2\,y = \frac{\partial(2\,xy)}{\partial x},\quad \frac{\partial(2\,xy)}{\partial z} = 0 = \frac{\partial(2\,xz)}{\partial y},$$

$$\frac{\partial(2\,xz)}{\partial x} = 2\,z = \frac{\partial(x + y^2 + z^2 + 1)}{\partial z}.$$

Hence the expression is an exact differential. Rearranging the terms,

$$du = (x + 1)dx + (y^2\,dx + 2\,xy\,dy) + (z^2\,dx + 2\,xz\,dz).$$

$$\therefore\ u = \frac{x^2}{2} + x + xy^2 + xz^2 + c.$$

EXERCISES

Which of the following differential expressions are exact differentials? If possible, find the functions of which they are the differentials.

1. $(2\,x + y)dx + (x + 2\,y)dy.$

2. $(2\,x - y)dx + (x - 2\,y)dy.$

3. $\dfrac{ydx - xdy}{x^2 + y^2}.$

4. $\dfrac{y}{x}dx + \log x dy.$

5. $\left(\sin\dfrac{y}{x} - \dfrac{y}{x}\cos\dfrac{y}{x}\right)dx + \cos\dfrac{y}{x}dy.$

6. $(y + z)dx + (z + x)dy + (x + y)dz.$

7. $(x^2 + y^2)dx + xydy.$

8. $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy.$

9. $(e^x yz + ye^z)dx + (e^x z + xe^z)dy + (e^x y + xye^z)dz.$

10. $\dfrac{ydx - xdy}{x^2 - y^2}.$

11. $(x + y^2 + xz)dx + 2\,xydy + x^2dz.$

12. $\left(\dfrac{y}{x} + \dfrac{y^2}{x^2}\right)dx + \left(\log x - \dfrac{2\,y}{x}\right)dy.$

CHAPTER XIII

ENVELOPES

115. Envelope of a family of curves. Let $f(x, y, a)$ be a continuous, single-valued function of x, y, and a, having the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial a}$. If x and y are the rectangular coördinates of a variable point in the plane, and a is a constant,

$$f(x, y, z, a) = 0 \tag{1}$$

is the equation of some curve, dependent in some manner, such as shape, size, or position, upon the value of a. As a varies continuously, equation (1) represents a whole family of curves. Thus

$$(x - a)^2 + y^2 = r^2 \tag{2}$$

represents a family of circles of radius r, the circle corresponding to an assigned value of a having its center at the point $(a, 0)$. Again

$$\frac{x}{a} + ay = 1 \tag{3}$$

represents a family of straight lines making with the coördinate axes a triangle whose area equals $\frac{1}{2}$.

The quantity a, which is constant for each curve of (1) but, in varying, determines the family of curves, is known as a *parameter*, and equation (1) is said to *involve the parameter a*.

If each of the curves of the family (1) is tangent[1] to a definite curve in the plane, the latter is known as the *envelope of the family* (or part of the envelope, if there are several such curves). Thus it is clear that each circle of (2) is tangent to each of the lines $y = r$ and $y = -r$. Hence each of these lines is part of the envelope of the family (2). Again, since the area of the triangle formed by every tangent to the equilateral hyperbola $xy = k^2$ with the coördinate axes is equal to $2k^2$, each of the lines of (3) is tangent to $xy = \frac{1}{4}$. This hyperbola is the envelope of the family (3). As a general fact it may be noted that any curve is the envelope of its tangents.

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[1] Two curves are said to be tangent to each other at a point if their tangents at that point coincide.

Each point P on the envelope is a point of one of the curves C of the family (1) corresponding to a definite value of a.

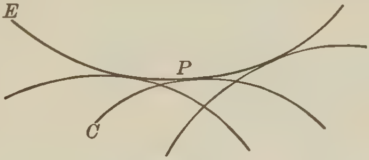

FIG. 106

The coördinates of a point on the envelope may thus be considered as functions of a. We shall suppose the equation of the envelope, in parametric form, to be

$$x = x(a), \quad y = y(a). \tag{4}$$

As a varies, P moves along the envelope E, in turn being momentarily a point on one or another of the curves of the family.

The slope of the envelope at P is the value of $\dfrac{\frac{dy}{da}}{\frac{dx}{da}}$ for the value of a determining P.

The slope of the curve C at P is the value of $-\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ for the same value of a, after x and y have been replaced by their values (4). Since these slopes are to be the same, we have $\dfrac{\frac{dy}{da}}{\frac{dx}{da}} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$; or, after obvious reductions,

$$\frac{\partial f}{dx}\frac{dx}{da} + \frac{\partial f}{\partial y}\frac{dy}{da} = 0. \tag{5}$$

Since P is on the curve C, the values of x and y given by (4) must satisfy equation (1), giving

$$f[x(a),\ y(a),\ a] = 0. \tag{6}$$

Differentiating (6) with respect to a,

$$\frac{\partial f}{\partial x}\frac{dx}{da} + \frac{\partial f}{\partial y}\frac{dy}{da} + \frac{\partial f}{\partial a} = 0. \tag{7}$$

Comparing this with (5), we see that

$$\frac{\partial f}{\partial a} = 0. \tag{8}$$

Solving equations (1) *and* (8) *for x and y, the equations* (4) *of the envelope are obtained.* The equation of the envelope in x and y may then be obtained by eliminating a between the equations (4); or more directly, *eliminating a between $f(x, y, a) = 0$ and $\frac{\partial f}{\partial a} = 0$ gives the equation of the envelope.* The following exercises will illustrate.

Ex. 1. Find the envelope of the family of circles

$$(x - a)^2 + y^2 = r^2.$$

Here

$$f \equiv (x - a)^2 + y^2 - r^2 = 0,$$

$$\frac{\partial f}{\partial a} \equiv -2(x - a) \qquad = 0.$$

Eliminating a, we have the equation of the envelope

$$y^2 - r^2 = 0, \quad \text{or} \quad y = \pm r.$$

Ex. 2. Find the envelope of the family of lines $\frac{x}{a} + ay = 1.$

Here
$$f \equiv \frac{x}{a} + ay - 1 = 0,$$
$$\frac{\partial f}{\partial a} \equiv -\frac{x}{a^2} + y \quad = 0.$$

Eliminating a, we have the equation of the envelope
$$xy = \tfrac{1}{4}.$$

At times, in actual practice, it is simpler to solve equations (1) and (8) for x and y, obtaining the equation of the envelope in the form (4). If desirable, the parameter may then be eliminated. The following exercise will illustrate.

Ex. 3. Find the envelope of a line of constant length l whose extremities move in the coördinate axes.

In Fig. 107, let AB be one of the positions of the moving line. Taking the equation of the moving line in the normal form
$$x \cos \alpha + y \sin \alpha = p,$$
evidently
$$p = OA \cos \alpha \text{ and } OA = l \sin \alpha.$$
$$\therefore p = l \sin \alpha \cos \alpha.$$

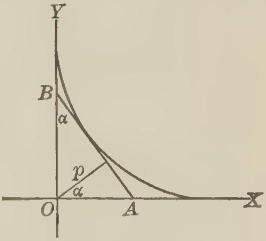

FIG. 107

We have then
$$f \equiv x \cos \alpha + y \sin \alpha - l \sin \alpha \cos \alpha = 0,$$
$$\frac{\partial f}{\partial \alpha} \equiv -x \sin \alpha + y \cos \alpha - l(\cos^2 \alpha - \sin^2 \alpha) = 0.$$

Instead of eliminating α directly, it is simpler in this case to solve these equations first for x and y and thus obtain the equations of the envelope in parametric form
$$x = l \cos^3 \alpha, \quad y = l \sin^3 \alpha.$$

Eliminating a between these, we have, finally,

$$\left(\frac{x}{l}\right)^{\frac{2}{3}}+\left(\frac{y}{l}\right)^{\frac{2}{3}}=1, \quad \text{or} \quad x^{\frac{2}{3}}+y^{\frac{2}{3}}=l^{\frac{2}{3}}.$$

Remark 1. — Attention should be called to the fact that there are certain restrictions on the form of the equation of the family of curves that can be used in finding the envelope. A discussion is beyond the scope of this book. As a simple illustration, let the equation of the family of circles (2) be written

$$f(x, y, a) \equiv x - a \pm \sqrt{r^2 - y^2} = 0.$$

From this form the relation (8),

$$\frac{\partial f}{\partial a} \equiv -1 = 0,$$

indicates the absence of an envelope. This is due to the form of $f(x, y, a)$.

Remark 2. — At times, the result of eliminating (a) between (1) and (8) may not be the equation of the envelope, or may involve extraneous factors, which when equated to zero prove to be curves which are not tangent to the curves of the family. These cases may arise when the curves of the family have singular points, such as double points or cusps, where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. In this case equation (5) is satisfied for any values of $\frac{dx}{da}$ and $\frac{dy}{da}$, and the argument used in obtaining the equation of the envelope fails. Without carrying out the discussion in this case, it may be noted that, whenever there is a locus of double points or cusps of the various curves of the family, this locus appears, along with any possible envelope, in the result of eliminating a between (1) and (8). As an illustration may be mentioned the family of cubics:

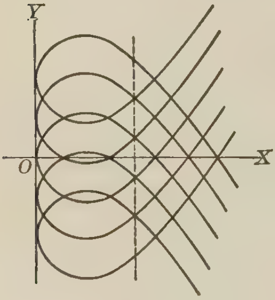

Fig. 108

$$f \equiv (y - a)^2 - x(x - 1)^2 = 0.$$

$$\frac{\partial f}{\partial a} \equiv -2\,a(y - a) \qquad = 0.$$

Eliminating a leads to

$$x(x - 1)^2 = 0, \text{ or } x = 0 \text{ and } x = 1.$$

A glance at Fig. 108 shows $x = 0$ to be an envelope, but $x = 1$ is the locus of the double points of the curves, and is not a part of the envelope.

116. Envelope of a family of curves involving two parameters connected by a relation. The problem of finding the envelope of the family of curves

$$f(x, y, a, b) = 0, \tag{1}$$

where the two parameters a and b are connected by the relation

$$\phi(a, b) = 0, \tag{2}$$

is not distinct in principle from that considered in the previous section. For one of the parameters may be obtained as a function of the other from (2); putting this value in (1), one parameter alone remains. But in practice, a more symmetrical way of handling the parameters will be found preferable at times:

Temporarily considering b as a function of a, we have

$$\left(\frac{\partial f}{\partial a}\right)_{a,b} \equiv \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{db}{da} = 0; \tag{3}$$

and from (2),

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b}\frac{db}{da} = 0. \tag{4}$$

Eliminating $\frac{db}{da}$ between (3) and (4), we have

$$\frac{\partial f}{\partial a}\frac{\partial \phi}{\partial b} - \frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a} = 0. \tag{5}$$

The envelope may be obtained by eliminating a, b, and $\frac{db}{da}$ between (1), (2), (3), and (4); or what is the same thing, by eliminating a and b between (1), (2), and (5). Thus, as a first exercise, we shall consider the third illustrative exercise of the previous section.

Ex. 1. Find the envelope of a line of constant length l, whose extremities move in the coördinate axes.

If the intercepts of the line are a and b, we have

$$f \equiv \frac{x}{a} + \frac{y}{b} - 1 = 0,$$

$$\phi \equiv a^2 + b^2 - l^2 = 0.$$

$$\therefore\ \frac{\partial f}{\partial a}\frac{\partial \phi}{\partial b} - \frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a} \equiv -2\left(\frac{bx}{a^2} - \frac{ay}{b^2}\right) = 0, \text{ or } \frac{x^{\frac{1}{3}}}{a} = \frac{y^{\frac{1}{3}}}{b}.$$

By composition

$$\frac{x^{\frac{1}{3}}}{a} = \frac{y^{\frac{1}{3}}}{b} = \frac{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}{\sqrt{a^2 + b^2}} = \frac{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}{l}.$$

$$\therefore\ \frac{x}{a} = \frac{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}{l},\ \frac{y}{b} = \frac{y^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}{l}, \text{ and}$$

$$\frac{x}{a} + \frac{y}{b} = \frac{(x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}}}{l} = 1, \text{ or}$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}.$$

Ex. 2. Find the envelope of the family of circles whose diameters are double ordinates of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The centers of the circles are evidently on the x-axis. Let the coördinates of the center of one of the circles be $(\alpha, 0)$ and its radius r. The equation of the family circles is then

$$f \equiv (x - \alpha)^2 + y^2 - r^2 = 0,$$

where α and r are two variable parameters, subject to the relation

$$\phi \equiv \frac{\alpha^2}{a^2} + \frac{r^2}{b^2} - 1 = 0.$$

Here $\frac{\partial f}{\partial \alpha}\frac{\partial \phi}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial \phi}{\partial \alpha} = -4(x - \alpha)\frac{r}{b^2} + \frac{4\,r\alpha}{a^2}$. Equating this to zero, and neglecting the special case $r = 0$, we have

$$\alpha = \frac{a^2 x}{a^2 + b^2}.$$

Using this in $\phi = 0$, we obtain

$$r^2 = b^2 - \frac{a^2b^2x^2}{(a^2+b^2)^2}.$$

Putting these in $f = 0$, we have, after simple reductions,

$$\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1,$$

the required equation of the envelope. It is another ellipse with the same minor axis as the original ellipse, but a larger major axis.

As a second method of solving this exercise, we take the equation of the given ellipse in parametric form

$$x = a \cos\theta, \quad y = b \sin\theta.$$

Then the equation of the family of circles is

$$(x - a\cos\theta)^2 + y^2 = b^2 \sin^2\theta,$$

involving the single parameter θ. Differentiating, we have

$$2\,a \sin\theta (x - a\cos\theta) = 2\,b^2 \sin\theta \cos\theta.$$

Neglecting the special case $\sin\theta = 0$, we have

$$ax - a^2\cos\theta = b^2\cos\theta, \text{ or } \cos\theta = \frac{ax}{a^2+b^2}.$$

$$\therefore \sin^2\theta = 1 - \frac{a^2x^2}{(a^2+b^2)^2}.$$

Putting these in the equation of the circle, we have

$$\left(1 - \frac{a^2}{a^2+b^2}\right)^2 x^2 + y^2 = b^2 - \frac{a^2b^2x^2}{(a^2+b^2)^2}.$$

After simple reductions, this leads to

$$\frac{x^2}{(a^2+b^2)^2} + \frac{y^2}{b^2} = 1.$$

EXERCISES

In the following exercises, either the method of the previous section, or that of this one is to be used, whichever seems the better adapted. When practicable, a sketch of the family of curves and of the envelope should be made.

1. Find the envelopes of the following families of lines, or, in other words, find the equations of the curves whose tangents have the following equations:

(*a*) $x \cos \alpha + y \sin \alpha - p = 0$, $p =$ constant.

(*b*) $y = 3\,ax + a^3$.

(*c*) $y = 2\,ax + a^4$.

(*d*) $y = ax + \dfrac{k}{a}\cdot$

(*e*) $y = mx + \sqrt{a^2m^2 \pm b^2}$, where m is the variable parameter.

(*f*) $y = \dfrac{x}{a} + a^2$.

2. Find the envelope of the family of lines whose intercepts on the axes have a constant sum k.

3. Find the envelope of the family of parabolas

(*a*) $y^2 = a(x - a)$.

(*b*) $y = a(x - a)^2$.

(*c*) $x^2 + a(x - 3\,y) + a^2 = 0$.

4. In § 40 the path of a projectile with initial speed v_0 and angle of fire α was found to be the parabola

$$y = -\frac{gx^2 \sec^2 \alpha}{2\,v_0^2} + x \tan \alpha.$$

What is the envelope of all possible paths for a given v_0 and variable α?

5. Find the envelope of the family of circles

(*a*) whose diameters are double ordinates of the parabola $y^2 = 4\,ax$;

(*b*) whose radii are equal to the constant b and whose centers lie on the circle $x^2 + y^2 = a^2$;

(*c*) which pass through the origin and have their centers on the equilateral hyperbola $x^2 - y^2 = a^2$;

(*d*) which pass through the origin and have their centers on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

(*e*) which pass through the origin and have their centers on the parabola $y^2 = 4\,ax$;

(*f*) whose diameters are the radii of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \text{or} \quad x = a\cos\theta, \quad y = b\sin\theta.$$

(*g*) $(x-a)^2 + y^2 = a$.

6. Find the envelope of the family of ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

if (*a*) $ab = k^2$, a constant; (*b*) $a + b = k$; (*c*) $a^2 + b^2 = k^2$.

7. Find the envelope of the family of equilateral hyperbolas $xy = a^2x + a$.

8. Find the envelope of the family of hyperbolas

$$a^2x^2 - ay^2 + k^2 = 0.$$

117. Second definition of envelope. Assigning to the parameter a a definite value a_1, a definite curve C_1 of the family is determined, its equation being

$$f(x, y, a_1) = 0. \tag{1}$$

Assigning to a a neighboring value $a_1 + \Delta a$, a second curve C_1',

$$f(x, y, a_1 + \Delta a) = 0 \tag{2}$$

is determined. These curves intersect in one or more points. In Fig. 109 one point of intersection M is pictured. The coördinates of the points of intersection C_1 and C_1' are obtained by solving (1) and (2) for x and y in terms of a_1 and Δa.

In this process, (2) may be replaced by the equation obtained by taking the difference between (1) and (2). Dividing by Δa, we have

$$\frac{f(x,\ y,\ a_1+\Delta a)-f(x,\ y,\ a_1)}{\Delta a}=0. \tag{3}$$

If now we make Δa smaller and smaller, the curve C_1' approaches C_1, and their point of intersection M approaches P. In the limit, as Δa approaches zero, equation (3) becomes

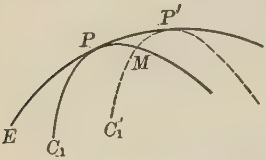

FIG. 109

$$\left[\frac{\partial f(x,\ y,\ a)}{\partial a}\right]_1=0, \tag{4}$$

where, using the notation of § 110, the left-hand member of (4) is the value of $\frac{\partial f(x,\ y,\ a)}{\partial a}$ for $a=a_1$. The coördinates of P, the limiting position of M, as Δa approaches zero, are obtained by solving (1) and (4) for x and y. These are found as functions of a_1, say

$$x=x(a_1),\ y=y(a_1). \tag{5}$$

If a_1 is no longer restricted to have a fixed value but is assigned any allowable value of a in $f(x,\ y,\ a)=0$, equations (5) define a curve, obtained by eliminating a_1 between these equations, or what is the same thing, by eliminating a_1 between (1) and (4). But we saw, § 115, that this elimination leads to the envelope of the family of curves $f(x,\ y,\ a)=0$. Hence the locus of the points P as defined here must also be the envelope. We shall understand by *ultimate points of intersection* of a curve of the family with a neighboring curve the limiting positions of the points of intersection as the neighboring curve approaches coincidence with the selected curve. Thus in Fig. 109, P is an ultimate point of intersection of the curve C_1 with a neighboring one, and P' is the ultimate point of intersection of the curve C' with a neighboring one. We may then define *the envelope of a family of curves* $f(x, y, a) = 0$ as *the locus of the ultimate points of intersection of each curve of the family with a neighboring one.*

118. Evolute of a curve. The envelope of the normals of a curve is known as the *evolute* of the curve.

The equation of the normal to a curve $f(x, y) = 0$ at the point (x_1, y_1) is

$$x - x_1 + y_1'(y - y_1) = 0. \tag{1}$$

Here x_1 may be taken as the parameter, y_1 and y_1' being functions of it. Differentiating with respect to the parameter

$$-1 + y_1''(y - y_1) - y_1'^2 = 0. \tag{2}$$

Solving (1) and (2) for x and y, we have the equation of the evolute in parametric form

$$x = x_1 - \frac{y_1'(1 + y_1'^2)}{y_1''}, \quad y = y_1 + \frac{1 + y_1'^2}{y_1''}. \tag{3}$$

Comparing these equations with (2), § 75, we see that *the evolute of a curve is the locus of the center of curvature of the curve.*

Ex. 1. Find the evolute of the parabola $y^2 = 4\,ax$.

The equation of the normal to this parabola at the point (x_1, y_1) is

$$x - x_1 + \frac{2\,a}{y_1}(y - y_1) = 0,$$

or

$$f \equiv y_1x - x_1y_1 + 2\,ay - 2\,ay_1 = 0. \tag{4}$$

The problem is to find the envelope of this family of lines with x_1 and y_1 as the parameters, subject to the relation

$$\phi \equiv y_1^2 - 4\,ax_1 = 0. \tag{5}$$

Here

$$\frac{\partial f}{\partial x_1}\frac{\partial \phi}{\partial y_1}-\frac{\partial f}{\partial y_1}\frac{\partial \phi}{\partial x_1}\equiv 4\,ax-4\,ax_1-2\,y_1^2-8\,a^2$$
$$=4\,a(x-3\,x_1-2\,a)=0. \quad (6)$$
$$\therefore x_1=\frac{x-2\,a}{3}.$$

Putting this in (4) and solving for y_1, we have

$$y_1=\frac{-3\,ay}{x-2\,a}.$$

Putting these in (5), we find the equation of the evolute to be

$$27\,ay^2=4(x-2\,a)^3.$$

Ex. 2. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the normal to this ellipse at the point (x_1, y_1) is
$$a^2y_1x-b^2x_1y=(a^2-b^2)x_1y_1.$$

Here two parameters x_1 and y_1 are involved subject to the relation
$$b^2x_1^2+a^2y_1^2=a^2b^2.$$

It will be found simpler in this case to make use of the parametric form of the equation of the ellipse and write $x_1=a\cos\theta$, $y_1=b\sin\theta$.

The equation of the normal may then be written

$$ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta.$$

To find its envelope, we differentiate with respect to θ.

$$ax\cos\theta+by\sin\theta=(a^2-b^2)(\cos^2\theta-\sin^2\theta).$$

Solving these two equations for ax and by we obtain

$$ax=(a^2-b^2)\cos^3\theta, \quad by=(a^2-b^2)\sin^3\theta.$$
$$\therefore (ax)^{\frac{1}{3}}=(a^2-b^2)^{\frac{1}{3}}\cos\theta, \quad (by)^{\frac{1}{3}}=(a^2-b^2)^{\frac{1}{3}}\sin\theta.$$

Squaring and adding we have the equation of the evolute

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

Ex. 3. Find the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

The equation of the normal at the point (x_1, y_1) is, as before,

$$x - x_1 + y_1'(y - y) = 0.$$

If θ is the value of the parameter corresponding to this point, $x_1 = a(\theta - \sin\theta)$, $y_1 = a(1 - \cos\theta)$, $y_1' = \dfrac{\sin\theta}{1 - \cos\theta}$. Hence the equation of the normal takes the following form after obvious simplification

$$x(1 - \cos\theta) + y\sin\theta = a\theta(1 - \cos\theta).$$

Differentiating with respect to θ

$$x\sin\theta + y\cos\theta = a(1 - \cos\theta) + a\theta\sin\theta.$$

Solving these for x and y we have the equation of the evolute in parametric form

$$x = a(\theta + \sin\theta),$$
$$y = -a(1 - \cos\theta).$$

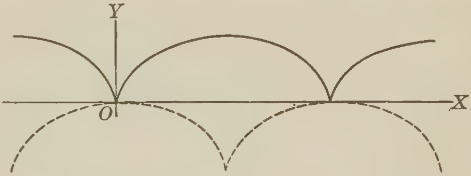

FIG. 110

This is another cycloid; the same curve in every respect as the original one except for its position. It is the dotted curve in Fig. 110. Its equation may be obtained from that of the original curve by moving the origin to the point $(a\pi, 2a)$ and replacing the parameter θ by $\theta + \pi$.

EXERCISES

Find the evolute of the following curves.

1. The equilateral hyperbola $xy = a^2$, or $x = at$, $y = \dfrac{a}{t}$.
2. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, or $x = a\cos^3\theta$, $y = a\sin^3\theta$.
3. The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, or $x = a\cosh u$, $y = b\sinh u$.
4. The curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

119. Envelope of a family of surfaces. A surface is said to be the *envelope* (or a part of it) *of a family of surfaces involving two parameters*

$$f(x, y, z, a, b) = 0 \tag{1}$$

if, at each point, it is tangent to a surface of the family. (Two surfaces are said to be tangent to each other at a point if their tangent planes at that point coincide.) Without going into details, we may say that the equation of the envelope of the family of surfaces may be found[1] by eliminating a and b between (1) and

$$\frac{\partial f}{\partial a} = 0, \tag{2}$$

$$\frac{\partial f}{\partial b} = 0. \tag{3}$$

Or, solving (1), (2), and (3) for x, y, and z in terms of a and b, we obtain the parametric equations of the surface.

Thus starting with the family of spheres of constant radius r and with their centers in the z-plane

$$f \equiv (x - a)^2 + (y - b)^2 + z^2 - r^2 = 0,$$

we have

$$\frac{\partial f}{\partial a} \equiv -(x - a) = 0, \quad \frac{\partial f}{\partial b} \equiv -(y - b) = 0.$$

Eliminating a and b, we have

$$z^2 = r^2 \quad \text{or} \quad z = \pm r.$$

It is obvious that either of these planes is tangent to each of the spheres of the family.

---

[1] As in the case of the envelope of a family of curves, attention should be called to the fact that there are certain restrictions on the form of the equation (1).

Also, if the surfaces (1) have singular points, the locus of these may appear in the result of the elimination of a and b between (1), (2), and (3), and must be allowed for.

A family of surfaces involving a single parameter

$$f(x, y, z, a) = 0 \tag{4}$$

also may have an envelope, which is found by eliminating a between (4) and

$$\frac{\partial f}{\partial a} = 0. \tag{5}$$

This family will frequently be selected from the larger family (1) involving two parameters, by giving some relation

$$\phi(a, b) = 0 \tag{6}$$

between the parameters. Considering b a function of a, the envelope is obtained by eliminating a, b, and $\frac{db}{da}$ between

$$f(x, y, z, a, b) = 0, \tag{1}$$

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{db}{da} = 0, \tag{7}$$

$$\phi(a, b) = 0, \tag{6}$$

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b}\frac{db}{da} = 0. \tag{8}$$

Or first eliminating $\frac{db}{da}$ between (7) and (8), obtaining

$$\frac{\partial f}{\partial a}\frac{\partial \phi}{\partial b} - \frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a} = 0, \tag{9}$$

the envelope may be obtained by eliminating a and b between (1), (6), and (9).

As an exercise let us find the envelope of those spheres of the family already considered, whose centers lie along the line $y = mx + h$. Here

$$f \equiv (x - a)^2 + (y - b)^2 + z^2 - r^2 = 0.$$

$$\phi \equiv b - ma - h = 0.$$

$$\frac{\partial f}{\partial a}\frac{\partial \phi}{\partial b} - \frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a} \equiv -2(x - a) - 2m(y - b) = 0$$

or

$$x - a + m(y - b) = 0.$$

Solving the first and last equations for $x - a$ and $y - b$, we find

$$y - b = \frac{\pm\sqrt{r^2 - z^2}}{\sqrt{m^2+1}}, \quad x - a = \frac{\mp m\sqrt{r^2 - z^2}}{\sqrt{m^2+1}}.$$

$$\therefore b = y \mp \frac{\sqrt{r^2 - z^2}}{\sqrt{m^2+1}}, \qquad a = x \pm \frac{m\sqrt{r^2 - z^2}}{\sqrt{m^2+1}}.$$

Substituting these in the second equation, we have, after obvious reductions,

$$(mx - y + h)^2 = (m^2+1)(r^2 - z^2).$$

This is the equation of a right circular cylinder whose axis is the line $y = mx + h$, as might have been expected.

EXERCISES

Find the envelopes of the following families of surfaces:

1. $ax + by + abz = 1$.

2. In Ex. 1 let (*a*) $a + b = 0$; (*b*) $a - b = 0$; (*c*) $ab = k$.

3. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where (*a*) $a + b + c = k$; (*b*) $abc = k$.

4. In each of Ex. 3 let (*a*) $a + b = 0$; (*b*) $a - b = 0$.

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where (*a*) $a + b + c = k$; (*b*) $abc = k^3$; (*c*) $a^2 + b^2 + c^2 = k^2$.

6. Spheres through the origin having their centers (*a*) on the elliptic paraboloid $x^2 + y^2 = 2z$; (*b*) on the hyperbolic paraboloid $x^2 - y^2 = 2z$.

CHAPTER XIV

MULTIPLE INTEGRALS

120. Double integral. We have had frequent occasion to make use of the definite integral

$$\int_{x_1}^{x_2} f(x)\,dx,$$

whose value is

$$F(x_2) - F(x_1),$$

where $F(x)$ is an indefinite integral of $f(x)$; that is,

$$\frac{dF(x)}{dx} = f(x), \text{ or } \int f(x)\,dx = F(x) + c.$$

Similarly we shall find very serviceable the *double integral*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dy\,dx. \tag{1}$$

Here x_1 and x_2 are constants, and either y_1 and y_2 may be a constant or a function of x. The double integral (1) involves two successive integrations:

First the integral

$$\int_{y_1}^{y_2} f(x, y)dy, \tag{2}$$

where x is considered constant, must be evaluated. We shall speak of it as a *partial definite integral with respect to y*. If $F(x, y)$ is an indefinite partial integral of $f(x, y)$ when x is constant, that is, $\frac{\partial F(x, y)}{\partial y} = f(x, y)$, the partial definite integral (2) equals

$$\int_{y_1}^{y_2} f(x, y)dy = F(x, y_2) - F(x, y_1). \tag{2}$$

This is a function of x only, and the value of the ordinary definite integral

$$\int_{x_1}^{x_2}[F(x, y_2) - F(x, y_1)]dx \tag{3}$$

is defined as that of the double integral (1).

Correspondingly the double integral

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)dx\,dy \tag{4}$$

is evaluated by first obtaining the partial definite integral with respect to x

$$\int_{x_1}^{x_2} f(x, y)dx = \Phi(x_2, y) - \Phi(x_1, y), \tag{5}$$

where $\Phi(x, y)$ is an indefinite integral of $f(x, y)$ when y is considered constant, and then finding the value of the definite integral

$$\int_{y_1}^{y_2}[\Phi(x_2, y) - \Phi(x_1, y)]dy. \tag{6}$$

In either case, the evaluation of the double integral involves that of two successive *simple* definite integrals, each involving a single variable, the other variable being considered constant in the first step of the process. The order of integration is important and may be remembered more readily if, at first, the integrals (1) and (4) are written

$$\int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x, y)dy\right]dx \quad \text{and} \quad \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x, y)dx\right]dy$$

respectively. Another way of writing the integrals is

$$\int_{x_1}^{x_2} dx\int_{y_1}^{y_2} f(x, y)dy \quad \text{and} \quad \int_{y_1}^{y_2} dy\int_{x_1}^{x_2} f(x, y)dx$$

respectively.

Some authors reverse the order of integration. While it is usually a very easy matter to tell from the appearance of the double integral what order is intended, it is well to bear this fact in mind.

Ex. 1. Evaluate

$$\int_1^2\int_{-3}^3(x^2-2\,xy)dy\,dx;\ \textit{i.e.}\ \int_1^2\left[\int_{-3}^3(x^2-2\,xy)dy\right]dx.$$

Considering x constant,

$$\int_{-3}^3(x^2-2\,xy)dy=x^2y-xy^2\Big]_{-3}^3=6\,x^2.$$

The value of the double integral is then

$$6\int_1^2 x^2\,dx=2\,x^3\Big]_1^2=14.$$

Ex. 2. Evaluate

$$\int_{-3}^3\int_1^2(x^2-2\,xy)dx\,dy;\ \textit{i.e.}\ \int_{-3}^3\left[\int_1^2(x^2-2\,xy)dx\right]dy.$$

Considering y constant,

$$\int_1^2(x^2-2\,xy)dx=\frac{x^3}{3}-x^2y\Big]_1^2=\frac{7}{3}-3\,y.$$

The value of the double integral is then

$$\int_{-3}^3\left(\frac{7}{3}-3\,y\right)dy=\frac{7\,y}{3}-\frac{3\,y^2}{2}\Big]_{-3}^3=14.$$

Note. It is no mere coincidence that the values of the double integrals in these two exercises are the same. It can be proved that *when $f(x, y)$ is continuous and x_1, x_2, y_1, y_2 are all constant,*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(x,\,y)dy\,dx=\int_{y_1}^{y_2}\int_{x_1}^{x_2}f(x,\,y)dx\,dy.$$

Ex. 3. Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}}(x+y)dy\,dx.$

Considering x constant,

$$\int_0^{\sqrt{a^2-x^2}}(x+y)dy=xy+\frac{y^2}{2}\Big]_0^{\sqrt{a^2-x^2}}=x\sqrt{a^2-a^2}+\frac{a^2-x^2}{2}.$$

The value of the double integral is then

$$\int_0^a\left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx=\frac{2a^3}{3}.$$

Ex. 4. Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a(1-\cos\theta)}\rho\sin\theta\,d\rho\,d\theta.$

Considering θ constant,

$$\int_0^{a(1-\cos\theta)}\rho\sin\theta\,d\rho=\frac{\rho^2\sin\theta}{2}\Big]_0^{a(1-\cos\theta)}=\frac{a^2}{2}(1-\cos\theta)^2\sin\theta,$$

the value of the double integral is then

$$\frac{a^2}{2}\int_0^{\frac{\pi}{2}}(1-\cos\theta)^2\sin\theta\,d\theta=\frac{a^2}{6}(1-\cos\theta)^3\Big]_0^{\frac{\pi}{2}}=\frac{a^2}{6}.$$

EXERCISES

Evaluate the following double integrals:

1. $\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2+y^2)dy\,dx.$

2. $\int_0^a\int_0^{\frac{\pi}{2}}\rho^3\,d\theta\,d\rho.$

3. $\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dy\,dx}{\sqrt{ax-x^2}}.$

4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)}\rho^2\sin\theta\,d\rho\,d\theta.$

121. Area as a double integral. Let $F(x, y)=0$ be a closed curve. To simplify matters, suppose that the curve is so shaped, that to each value of x there correspond not more than two values of y, $y_1=f_1(x)$ and $y_2=f_2(x)$, which become equal when $x=a$ and $x=b$, and to each value of y there correspond not more than two values of x, $x_1=\phi_1(y)$ and $x_2=\phi_2(y)$, which become equal when $y=\alpha$ and $y=\beta$.

Divide the interval (a, b) into n equal parts, each Δx in length, and draw a set of lines parallel to the y-axis through each of the points of division. Similarly, divide the interval (α, β) into m equal parts, each Δy in length, and draw a set of horizontal lines through each of the points of division. The area enclosed by $F(x, y) = 0$ is broken up into a number of rectangles, excepting along the boundary, where parts only of rectangles belong to the area. The sum of the areas of these complete and partial rectangles equals the area enclosed by the curve.

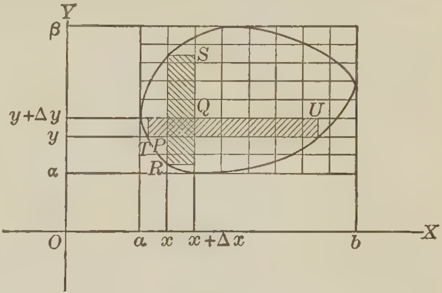

FIG. 111

To evaluate this sum, we first find the area of each vertical strip of rectangles, and then find the sum of the areas of all these strips. Thus, let (x, y) be a point P and $(x+\Delta x, y+\Delta y)$ the point Q, both lying inside the bounding curve. The area of the rectangle equals $\Delta y\,\Delta x$. The area of the vertical strip of complete and partial rectangles including PQ differs from the rectangle RS in that all the component rectangles of the latter are complete. Now

$$\text{Area } RS = \Delta x\sum\Delta y = \Delta x[f_2(x) - f_1(x)] = \Delta x\int_{y_1}^{y_2} dy.$$

Here, as usual, the Greek letter $\sum$, corresponding to our S, indicates that we are to take the sum of all the expressions of the type following it.

If we take $\sum \text{Area } RS = \sum\Delta x\int_{f_1(x)}^{f_2(x)} dy$, the sum of all the strips of complete rectangles, we obtain an area which differs from that of the area bounded by the curve, because in certain cases some of the partial rectangles along the boundary are neglected and in other cases they are replaced by complete rectangles. But it can be proved that, when Δx and Δy are allowed to approach zero, the sum of the areas of the strips

RS approaches the area enclosed by the curve. At the same time (§§ 88 and 120)

$$\underset{\Delta x \to 0}{L} \sum [f_2(x) - f_1(x)]\Delta x = \int_a^b [f_2(x) - f_1(x)]\, dx = \int_a^b \int_{y_1}^{y_2} dy\, dx.$$

Hence *the numerical value of the area bounded by the curve is equal to that of the double integral*

$$\int_a^b \int_{y_1}^{y_2} dy dx.$$

The *element of area in the double integral* is $dA = dy\, dx$.

We might equally well have first found the curve of each of the horizontal strips of rectangles, of which *TU* is a specimen, and then summed these. By this method the area bounded by the curve is found to be

$$\underset{\Delta y \to 0}{L} \sum [\phi_2(y) - \phi_1(y)]\Delta y = \int_\beta^\alpha [\phi_2(y) - \phi_1(y)] dy = \int_\alpha^\beta \int_{x_1}^{x_2} dx\, dy.$$

All this applies equally well in case the area to be found is enclosed by two or more curves, and also in case the boundary of the area is met by some horizontal or vertical lines in more than two points. The required modifications will suggest themselves in any particular case.

Attention should be called to the fact that the actual work involved in calculating as a double integral the area enclosed by one or more curves is identical with that required when the area is given as a simple definite integral (§ 93). In practice, the latter method is preferable. But the principles involved in the method considered in this section are of great value in other types of problems to be considered in this chapter.

122. Double integral as the limit of a sum. In § 88 we saw that a simple definite integral is the limit of a sum,

$$\underset{n \to \infty}{L} \sum_{i=0}^{n} f(x_i)(x_{i+1} - x_i) = \int_a^b f(x)\, dx.$$

In an analogous manner it can be shown that a double integral also is the limit of a sum. It is not feasible to give a proof of this fact here. But the following outline will suffice.

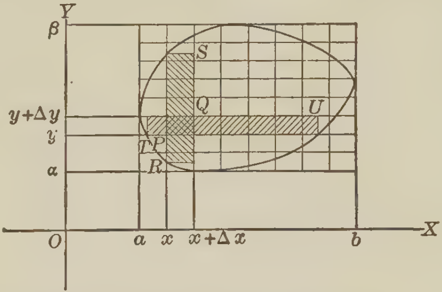

FIG. 111

Let $f(x, y)$ be a continuous function of x and y over A, an area bounded by one or more curves. As in § 121 we shall suppose this area divided into complete and partial rectangles as illustrated in Fig. 111. If, as before, the coördinates of P are (x, y) and those of Q are $(x + \Delta x, y + \Delta y)$, consider the expression

$$f(x, y)\,\Delta x\,\Delta y, \tag{1}$$

which is the product of the value of $f(x, y)$ at any point of the rectangle PQ by the area of that rectangle. Take the sum of these products for all the rectangles belonging to A,

$$\sum f(x, y)\,\Delta x\,\Delta y. \tag{2}$$

It can be proved that, *as Δx and Δy approach zero, the sum* (2) *approaches a definite limit which is designated by*

$$\iint\limits_{(A)} f(x, y)\,dy\,dx, \tag{3}$$

and is known as the double integral of the function $f(x, y)$ taken over the area A. It can be proved also that *the value of the double integral* (3) *is equal to that of either of the double integrals*

$$\int_a^b \int_{y_1}^{y_2} f(x, y)dy\,dx \text{ and } \int_\alpha^\beta \int_{x_1}^{x_2} f(x, y)\,dx\,dy, \tag{4}$$

where the notation for the limits is that used in § 121.

Moreover, if the area A is broken up into elementary areas ΔA, other than rectangles, according to *any* law, provided the greatest dimensions of each of the elementary areas approaches zero as their number grows indefinitely, it can be proved that the limit of the sum $\sum f(x, y)\Delta A$ approaches the same limit (3); *i.e.*

$$\iint\limits_{(A)} f(x, y)\, dA = \iint\limits_{(A)} f(x, y)\, dx\, dy. \tag{5}$$

123. Volume under a surface. Let $z = f(x, y)$ be the equation of a surface and $F(x, y) = 0$ the equation of a curve in the xy-plane enclosing an area A. The volume of the cylinder $F(x, y) = 0$ included between the xy-plane and the given surface will be referred to as "the volume under the surface," even if part or all the given surface lies below the xy-plane. To find this volume, we divide the area enclosed by the curve $F(x, y) = 0$ in the xy-plane into rectangles whose dimensions are Δx and Δy, and consider the elementary parallelopipeds whose volumes are

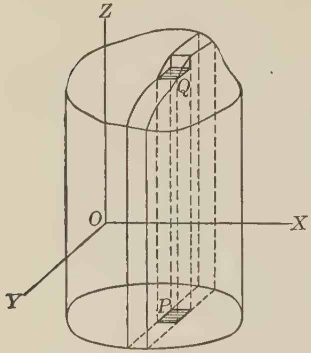

FIG. 112

$$\Delta V = z\,\Delta x\,\Delta y = f(x, y)\Delta x\,\Delta y,$$

of which one is pictured in Fig. 112. The limit of the sum

$$\sum f(x, y)\Delta x\,\Delta y$$

as Δx and Δy approach zero is equal to the required volume. But this double sum has for limit the double integral

$$\iint\limits_{(A)} f(x, y)\, dx\, dy.$$

Hence the value of the required volume is given by the formula

$$V = \int_a^b \int_{y_1}^{y_2} f(x, y) dy\, dx = \int_\alpha^\beta \int_{x_1}^{x_2} f(x, y) dx\, dy. \qquad (1)$$

The *element of volume* is $dV = f(x, y) dx\, dy$.

Remark. — It should be noted that the partial integral $\int_{y_1}^{y_2} f(x, y) dy$ is the area of the section of the solid by a plane parallel to the yz-plane at a distance x from the latter. If the area of this section is a known function of x, say $\phi(x)$, the required volume is given by the simple definite integral $\int_a^b \phi(x) dx$. Similarly the partial integral $\int_{x_1}^{x_2} f(x, y) dx$ is the area of the section of the solid by a plane parallel to the xz-plane at a distance y from the latter. If the area of this section is a known function of y, say $\psi(y)$, the required volume is given by the simple definite integral $\int_\alpha^\beta \psi(y) dy$. The method of § 98 makes use of this fact. Whenever it is possible to employ this method, it is shorter than the general method of this section, for it eliminates the first integration.

124. Area of curve in polar coördinates. The area bounded by one or more curves, given in polar coördinates, can be expressed as a double integral. To simplify matters we shall suppose there is a single bounding curve $\Phi(\rho, \theta) = 0$, which is convex; that is, no straight line cuts it in more than two points. Hence, if the pole is outside the curve, to each value of θ there correspond two values of ρ, $\rho_1 = f_1(\theta)$ and $\rho_2 = f_2(\theta)$, which become equal when $\theta = \theta_a$ and $\theta = \theta_b$; similarly, to each value of ρ there correspond two values of θ, $\theta_1 = \phi_1(\rho)$ and $\theta_2 = \phi_2(\rho)$, which become equal when $\rho = \rho_a$ and $\rho = \rho_b$. The curve $\Phi(\rho, \theta) = 0$ is tangent to the vectorial radii, making the angles θ_a and θ_b, respectively, with the initial line, and also to the circles with centers at the origin (or pole) and radii ρ_a and ρ_b respectively.

In Fig. 113, the angle AOB is divided into n equal parts, each $\Delta\theta$ in size. The length DA is divided into m equal parts, each equal to $\Delta\rho$, and arcs of circles, with centers at the origin, are drawn through the points of division. The area *ABCD* is thus divided into smaller areas of similar shape. We shall call them quadrilaterals. When n and m are large, the area bounded by $\Phi(\rho, \theta) = 0$ is made up of a large number of these quadrilaterals, together with partial ones along its boundary.

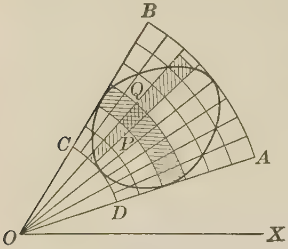

FIG. 113

Let (ρ, θ) be the coördinates of some point P inside the curve, and $(\rho + \Delta\rho, \theta + \Delta\theta)$ those of a neighboring point Q. The area of the quadrilateral PQ is

$$\Delta A = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta - \tfrac{1}{2}\rho^2\Delta\theta = \rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\Delta\rho^2\,\Delta\theta.$$

As n and m are made larger and larger, $\Delta\rho$ and $\Delta\theta$ get smaller and smaller. Since the second term of the right-hand member contains three of these small factors while the first term contains two of them only, the latter becomes a good approximation for ΔA when n and m are large. Moreover, letting $\sum\rho\,\Delta\rho\,\Delta\theta$ represent the sum of the complete quadrilaterals inside $\Phi(\rho, \theta) = 0$ for a given choice of n and m, it can be proved that the limit of this sum, as n and m approach infinity, that is,

$$\underset{\substack{n \to \infty \\ m \to \infty}}{L} \sum \rho\,\Delta\rho\,\Delta\theta,$$

is actually equal to the required area. But this limit is precisely the double integral

$$\iint\limits_{(A)} \rho\,d\rho\,d\theta$$

taken over the portion of the plane enclosed by $\Phi(\rho, \theta) = 0$. According as the quadrilaterals are summed for ρ first and then for θ or in the reverse order, we have

$$\text{Area} = \int_{\theta_a}^{\theta_b}\int_{\rho_1}^{\rho_2} \rho \, d\rho \, d\theta = \int_{\rho_a}^{\rho_b}\int_{\theta_1}^{\theta_2} \rho \, d\theta \, d\rho. \qquad (1)$$

The *element of area* is $dA = \rho \, d\rho \, d\theta$.

If the pole is inside the bounding curve, formula (1) still holds. In this case the lower limit for ρ is 0, and those for θ, in the first integral at least, are 0 and 2π.

Attention should be called to the fact that the work involved in evaluating the first integral in (1) is precisely that required when the area is given as a simple definite integral (§ 94). In practice, the latter method is usually preferable. At times it may be simpler to sum for θ first. This is true especially in case the length of arc of the circle with center at O and radius ρ intercepted by the bounding curve or curves is readily expressed as a function of ρ, say $F(\rho)$. For in this case the first partial integral $\int_{\theta_1}^{\theta_2} \rho \, d\theta = F(\rho)$; and the area is given by the simple integral $\int_{\rho_a}^{\rho_b} F(\rho) d\rho$. As a very simple example, but one of considerable practical importance, may be mentioned the problem of finding the area of a circle of radius a.

$$\text{Area} = \int_0^a \int_0^{2\pi} \rho \, d\theta \, d\rho = 2\pi \int_0^a \rho \, d\rho = \pi a^2.$$

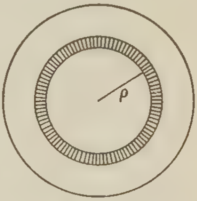

FIG. 114

Stopping with the first partial integration, we see that we may take $2\pi\rho \, d\rho$ as the *element of area*. It is approximately the area of the shaded ring in Fig. 114. It is *the differential of the area of the circle*, when ρ is taken as the variable of reference. This will explain the rather striking fact that the derivative of the area of a circle πr^2 equals $2\pi r$, the circumference of the circle.

125. Volume under a surface in cylindrical coördinates. At times it is simpler to express the equation of the cylinder bounding the volume under consideration in polar coördinates. Let it be $\Phi(\rho, \theta) = 0$. In such a case, it is convenient to use the *cylindrical coördinates* (ρ, θ, z) to represent the points of the surface. Here we still use the rectangular coördinate z for the distance of the point from the xy-plane, but we use the polar coördinates ρ and θ of the projection of this point on the xy-plane instead of its rectangular coördinates. The equation of the surface in cylindrical coördinates may be written in the form

$$z = f(\rho, \theta).$$

In rectangular coördinates, we saw that the element of volume is the product of z by the element of area in the plane $dx\,dy$; *i.e.*

$$dV = z\,dx\,dy.$$

In an entirely analogous manner, it can be shown that the element of volume in cylindrical coördinates is the product of z by the element of area in the plane $\rho\,d\rho\,d\theta$; *i.e.*

$$dV = z\rho\,d\rho\,d\theta = \rho f(\rho, \theta) d\rho\,d\theta.$$

Hence the volume under the surface is

$$V = \int_{\theta_a}^{\theta_b}\int_{\rho_1}^{\rho_2} \rho f(\rho, \theta) d\rho\,d\theta = \int_{\rho_a}^{\rho_b}\int_{\theta_1}^{\theta_2} \rho f(\rho, \theta) d\theta\,d\rho. \qquad (1)$$

Ex. 1. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $\rho = a \cos\theta$.

The equation of the sphere in cylindrical coördinates is $z = \pm\sqrt{a^2 - \rho^2}$. The required volume is twice that under the surface $z = \sqrt{a^2 - \rho^2}$, bounded by the given cylinder.

$$\therefore V = 2\iint_{(A)} \rho\sqrt{a^2 - \rho^2}\,d\rho\,d\theta.$$

If the first partial integral is taken with respect to ρ, the limits are 0 and $a\cos\theta$, respectively. Then the final integral for θ has the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, respectively; for, as the radius vector sweeps over this range, the entire circle $\rho = a\cos\theta$ is traced. Evidently, from geometrical considerations, this final integral is equal to twice the corresponding integral in which the limits are 0 and $\frac{\pi}{2}$, respectively. Hence we have

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\rho\sqrt{a^2-\rho^2}\,d\rho\,d\theta.$$

$$\int_0^{a\cos\theta}\rho\sqrt{a^2-\rho^2}d\rho = -\tfrac{1}{3}(a^2-\rho^2)^{\frac{3}{2}}\Big]_0^{a\cos\theta} = \frac{a^3}{3}(1-\sin^3\theta).$$

$$\therefore\ V = \frac{4\,a^3}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta = \frac{4\,a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

Ex. 2. Find the volume between the sphere $x^2+y^2+z^2=9$ and the paraboloid $x^2+y^2=8z$.

The equations of the surfaces in cylindrical coördinates are $\rho^2+z^2=9$ and $\rho^2=8z$, respectively. The surfaces intersect in a horizontal circle whose radius and height above the xy-plane are the common roots of $\rho^2+z^2=9$ and $\rho^2=8z$. The equation

$$z^2+8z-9=0$$

has the two roots 1 and -9. The second of these has no interest for us, since it corresponds to an imaginary intersection of the two surfaces. Corresponding to $z=1$, we have $\rho=2\sqrt{2}$.

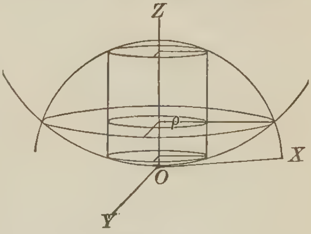

FIG. 115

Letting z_2 and z_1 be the values of z corresponding to a given value of ρ in the equations of the sphere and paraboloid, respectively, the required volume is given by

$$V = \int_0^{2\sqrt{2}}\int_0^{2\pi}(z_2-z_1)\rho\,d\theta\,d\rho = 2\,\pi\int_0^{2\sqrt{2}}(z_2-z_1)\rho\,d\rho.$$

In case the solid whose volume is sought is one of revolution, as it is in this case, the volume can always be obtained by a single integration, the element of volume being that of a hollow cylinder whose altitude is $z_2 - z_1$ and the area of whose base is taken to be $2\pi\rho\, d\rho$. Compare method of § 96.

$$\therefore\ V = 2\pi\int_0^{2\sqrt{2}}\left(\rho\sqrt{9-\rho^2} - \frac{\rho^3}{8}\right)d\rho = \frac{40\pi}{3}.$$

Ex. 3. Evaluate $\int_0^\infty e^{-x^2}\,dx$.

It is very easy to see from the rule for evaluating a double integral that

$$\int_a^b\int_\alpha^\beta f(x)\phi(y)dy\,dx = \int_a^b f(x)dx\int_\alpha^\beta \phi(y)dy.$$

Here both sets of limits are constants, and the integrand is a function of x only multiplied by a function of y only.

If we write $A = \int_0^\infty e^{-x^2}\,dx$, then evidently we may write also $A = \int_0^\infty e^{-y^2}\,dy$. Hence

$$A^2 = \int_0^\infty e^{-x^2}\,dx\int_0^\infty e^{-y^2}\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

The integral on the right may be looked upon as the volume under the surface $z = e^{-(x^2+y^2)}$, lying in the first octant. In cylindrical coördinates the equation of this surface is $z = e^{-\rho^2}$, and the volume under it in the first octant is

$$A^2 = \int_0^\infty\int_0^{\frac{\pi}{2}} \rho e^{-\rho^2}\,d\theta\,d\rho = \frac{\pi}{4}.$$

$$\therefore\ A = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

EXERCISES

Whenever practicable reduce the problem to the evaluation of a simple definite integral.

1. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = b^2$, where $0 < b < a$.

2. Find the volume between the sphere $x^2 + y^2 + z^2 - 2\,az = 0$ and the paraboloid $x^2 + y^2 = az$.

3. Find the volume between the sphere $x^2 + y^2 + z^2 = 8$ and the cone $x^2 + y^2 - z^2 = 0$.

4. Find the volume between the sphere $x^2 + y^2 + z^2 - 2\,az = 0$ and the cone $x^2 + y^2 - z^2 = 0$.

5. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder whose directrix is one loop of $\rho^2 = a^2 \cos^5 2\,\theta$.

126. Area of a surface. Let A be the area enclosed by one or more curves lying in a plane MP. If each point on the boundary of this area is projected vertically upon a second plane MP', making an angle θ with the first plane, the aggregate of the points of projection encloses an area A' whose value is

$$A' = A \cos \theta. \qquad (1)$$

To show this, divide the area A into rectangles by drawing a set of lines in the plane MP parallel and perpendicular to MN, the line of intersection of the two planes. The projection of these sets of lines upon the plane MP' is two sets of lines, also parallel and perpendicular to MN, which divide A' into rectangles. The sides a and b of

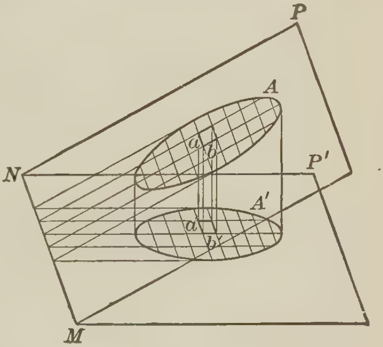

FIG. 116

a rectangle of A, parallel and perpendicular respectively to MN, are projected into the sides a' and b' of the corresponding rectangle of A', and besides

$$a' = a, \quad b' = b \cos \theta.$$
$$\therefore \; a'b' = ab \cos \theta.$$

When the number of rectangles is made larger and larger, with a and b each approaching zero as a limit, we have for the limit of the sum of the rectangles making up the area A'

$$\underset{\substack{a' \to 0 \\ b' \to 0}}{L} \sum a'b' = \underset{\substack{a \to 0 \\ b \to 0}}{L} \sum ab \cos \theta = \cos \theta \underset{\substack{a \to 0 \\ b \to 0}}{L} \sum ab.$$

As can be seen from § 121, this is precisely

$$A' = A \cos \theta. \tag{1}$$

If now it is desired to find an expression for the area of a part of a surface bounded by one or more curves we proceed as follows:

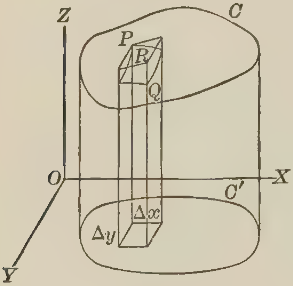

FIG. 117

Let C be the boundary of the part of the surface whose area is sought, and C' the projection of this boundary upon the xy-plane. Divide the area bounded by C' into rectangles of dimensions Δx and Δy by lines parallel to the axes of x and y respectively. Through these lines pass planes parallel to the z-axis which will divide the area, S, of the part of the surface bounded by C into elementary areas ΔS. The required area is the sum of these elementary areas $\sum \Delta S$. Let PQ be a typical elementary area. The vertical planes, which cut out the latter, will cut out a parallelogram from the tangent plane to the surface at some point in PQ such as P. Let us call it $\Delta_1 S$. In Fig. 117 this parallelogram is PR. It can be proved that the limit of $\sum \Delta_1 S$ as Δx and

Δy approach zero exists, and is independent of the choice of coördinates or of the way in which the elementary areas in C' are selected. This limit is taken as the value of the required area. But we saw above, formula (1), that

$$\Delta x\, \Delta y = \Delta_1 S \cos\gamma, \text{ whence } \Delta_1 S = \frac{\Delta x\, \Delta y}{\cos\gamma},$$

where γ is the angle between the tangent plane at P and the xy-plane. This is precisely the angle which the normal to the surface at P makes with the z-axis. If the equation of the surface is $F(x, y, z) = 0$, the direction cosines of the normal at (x, y, z) are proportional to $\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}$, and

$$\cos\gamma = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}.$$

$$\therefore \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{L} \sum \Delta_1 S = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{L} \sum \frac{\Delta x\, \Delta y}{\cos\gamma}$$

$$= \iint_{(C')} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dy}{\frac{\partial F}{\partial z}}.$$

Hence the required area is given by the formula

$$S = \iint_{(C')} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dy}{\frac{\partial F}{\partial z}}. \tag{2}$$

Letting dA' be the element of area of C', this formula can be written in the more compact form

$$S = \iint_{(C')} \frac{dA'}{\cos\gamma}. \tag{2'}$$

There is also a practical advantage in the form of (2′). For at times it is simpler to evaluate the double integral by using polar coördinate in the xy-plane, in which case $dA' = \rho\, d\rho\, d\theta$.

At times it is preferable for practical purposes in evaluating the double integral, to project the curve C upon the yz-plane or upon the xz-plane. Calling the projections of C upon these planes C'' and C''' respectively, and dA'' and dA''' their respective elements of area, the area is given by

$$S = \int\!\!\int_{(C'')} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dy\, dz}{\frac{\partial F}{\partial x}} = \int\!\!\int_{(C'')} \frac{dA''}{\cos\alpha} \qquad (2'')$$

or

$$S = \int\!\!\int_{(C''')} \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dz}{\frac{\partial F}{\partial y}} = \int\!\!\int_{(C''')} \frac{dA'''}{\cos\beta}. \qquad (3''')$$

Ex. 1. Find the area of the part of the upper hemisphere $x^2 + y^2 + z^2 = a^2$(i.e. $z = \sqrt{a^2 - x^2 - y^2}$) cut out by the cylinder $x^2 + y^2 - ax = 0$.

Here

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 2z. \quad \therefore \cos\gamma = \frac{z}{a} = \frac{\sqrt{a^2 - x^2 - y^2}}{a}.$$

C' is the circle $x^2 + y^2 - ax = 0$. Integrating for y first, the limits are $-\sqrt{ax - x^2}$ and $\sqrt{ax - x^2}$. The limits for x are 0 and a. Hence

$$S = a\int_0^a \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \frac{dy\, dx}{\sqrt{a^2 - x^2 - y^2}} = 2a\int_0^a \int_0^{\sqrt{ax-x^2}} \frac{dy\, dx}{\sqrt{a^2 - x^2 - y^2}}.$$

The work involved in evaluating this is simplified if we use cylindrical coördinates. Then, since the polar equation of C' is $\rho = a\cos\theta$,

$$S = 2a\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \frac{\rho\, d\rho\, d\theta}{\sqrt{a^2 - \rho^2}} = a^2(\pi - 2).$$

Ex. 2. Find the area of the part of the cylinder $x^2 + y^2 - ax = 0$ between the xy-plane and the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

It is evidently simplest to use formula (2‴) in this case. The desired area is twice that in the first octant.

$$\frac{\partial F}{\partial x} = 2x - a, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 0. \quad \therefore \cos\beta = \frac{2y}{a}.$$

$$\therefore S = a \iint\limits_{(C''')} \frac{dx\,dz}{y} = a \iint\limits_{(C''')} \frac{dx\,dz}{\sqrt{ax - x^2}}.$$

Integrating for z first, the limits are 0 and $\sqrt{a^2 - x^2 - y^2}$. But, since for points in the surface (cylinder) the coördinates are connected by the relation $x^2 + y^2 = ax$, the limits for z are 0 and $\sqrt{a^2 - ax}$. The limits for x are 0 and a.

$$\therefore S = a \int_0^a \int_0^{\sqrt{a^2 - ax}} \frac{dz\,dx}{\sqrt{ax - x^2}} = a \int_0^a \frac{\sqrt{a^2 - ax}}{\sqrt{ax - x^2}}\,dx$$

$$= a^{\frac{3}{2}} \int_0^a \frac{dx}{\sqrt{x}} = 2a^2.$$

EXERCISES

1. Find the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 2az$ cut out by the paraboloid $x^2 + y^2 = z$.

2. Find the area of the entire portion of the surface of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = mx$.

3. Find the area of the portion of the plane $x + y + z = 2a$ in the first octant bounded by the cylinder $x^2 + y^2 = a^2$.

4. Find the area of the portion of the surface of the cylinder $x^2 + y^2 = a^2$ lying in the first octant and under the plane $x + y + z = 2a$.

5. Find the area of the portion of the surface of the cylinder $x^2 + z^2 = a^2$ in the first octant cut out by the cylinder $x^2 + y^2 = a^2$.

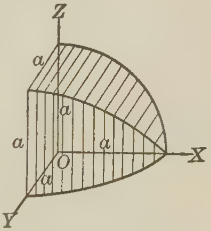

FIG. 118

6. Find the area of the portion of the surface of the paraboloid $y^2 + z^2 - 4ax = 0$ intercepted by the cylinder $y^2 = ax$ and the plane $x = 3a$.

7. Find the area of the portion of the surface of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ outside the cylinder whose directrix is $\rho = a \sin 2\theta$.

127. Triple integral. Analogous to the double integral we have the *triple integral*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)dz\,dy\,dx, \tag{1}$$

which involves three successive integrations:

First the partial integral with respect to z,

$$\int_{z_1}^{z_2} f(x, y, z)dz,$$

where x and y are considered constant, must be evaluated. Here z_1 and z_2 are constants, or functions of either x or y, or of both. If $F(x, y, z)$ is an indefinite integral of $f(x, y, z)$ when x and y are constant, that is, $\dfrac{\partial F(x, y, z)}{\partial z} = f(x, y, z)$,

$$\int_{z_1}^{z_2} f(x, y, z)dz = F(x, y, z_2) - F(x, y, z_1).$$

This is a function of x and y only. The value of the double integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x, y, z_2) - F(x, y, z_1)]dy\,dx$$

(which involves two successive integrations) is that of the triple integral (1). The order of the integrations involved in (1) may be emphasized by writing it in the form

$$\int_{x_1}^{x_2}\left[\int_{y_1}^{y_2}\left\{\int_{z_1}^{z_2} f(x, y, z)dz\right\}dy\right]dx.$$

EXERCISES

Evaluate the following triple integrals:

1. $\int_0^a \int_0^x \int_0^{\sqrt{x^2-y^2}} (x^2+y^2)dz\,dy\,dx.$

The first partial integration with respect to z gives

$$\int_0^{\sqrt{x^2-y^2}} (x^2+y^2)dz = (x^2+y^2)\sqrt{x^2-y^2}.$$

Then integrating with respect to y,

$$\int_0^x (x^2\sqrt{x^2-y^2}+y^2\sqrt{x^2-y^2})dy = \frac{5\,\pi x^4}{16}.$$

Finally $$\frac{5\,\pi}{16}\int_0^a x^4\,dx = \frac{\pi a^5}{16}.$$

2. $\int_0^3 \int_0^1 \int_0^2 (x+y-z)dz\,dy\,dx.$

3. $\int_0^a \int_0^z \int_0^{y+z} xyz\,dx\,dy\,dz.$

4. $\int_0^\pi \int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \rho^2 \sin\theta\,d\rho\,d\theta\,d\phi.$

128. Volume as a triple integral. Let $F(x, y, z)=0$ be a closed surface, which, to simplify matters, will be supposed to be convex; that is, every plane section of the surface is a convex curve. Suppose that the surface is enclosed by the planes $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$, $z=c_1$, $z=c_2$, which are tangent to it.

Divide the interval (a_1, a_2) into n equal parts, each Δx in length, and draw a set of planes parallel to the yz-plane through each of the points of division and the end points. Similarly, divide the interval (b_1, b_2) into m equal parts, each Δy in length, and draw a set of planes parallel to the xz-plane. Finally, divide the interval (c_1, c_2) into l equal parts, each Δz in length, and draw a set of planes parallel to the xy-plane. The solid enclosed by $F(x, y, z)=0$ is broken up into a number of parallelopipeds, excepting along the boundary, where parts only of the parallelopiped belong to the solid. The sum of the volumes of these complete and partial parallelopipeds equals the volume of the enclosed solid.

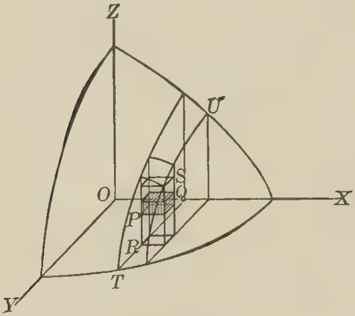

Fig. 119

To evaluate this sum, we first find the volume of each vertical column of parallelopipeds; then find the volume of each slice of columns parallel to the yz-plane; finally find the sum of all these slices.

In order to avoid too complicated a diagram, Fig. 119 pictures the part of the surface lying in the first octant.

Starting with a parallelopiped PQ, the coördinates of two opposite vertices being (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$, consider the volume cut from the solid by the planes of the vertical faces of PQ. It differs from the parallelopiped RS in that all the component parallelopipeds of the latter are complete.

$$\text{Vol } RS = \Delta x \, \Delta y \sum \Delta z = \Delta x \, \Delta y \, (z_2 - z_1) = \Delta x \, \Delta y \int_{z_1}^{z_2} dz,$$

where z_1 and z_2 are the two values of z given as functions of the x and y of P when $F(x, y, z) = 0$ is solved for z.

The sum of the volumes of the parallelopipeds in the same slice with RS differs from the volume of the corresponding slice of the solid TU in the fact that the part of the boundary of the latter supplied by the surface $F(x, y, z) = 0$ is replaced by a set of horizontal and vertical planes. This sum will be represented by

$$\sum \text{Vol } RS = \Delta x \sum (z_2 - z_1) \Delta y = \Delta x \sum \left[\int_{z_1}^{z_2} dz \right] \Delta y.$$

Finally, the sum of these sums is

$$\sum\sum \text{Vol } RS = \sum\Big[\sum (z_2 - z_1)\Delta y\Big]\Delta x = \sum\left\{\sum\left[\int_{z_1}^{z_2} dz\right]\Delta y\right\}\Delta x.$$

It can be proved that as the Δx, Δy, Δz approach zero as a limit, $\sum$ Vol RS approaches Vol TU as a limit, and $\sum\sum$ Vol RQ approaches the volume of the solid bounded by $F(x, y, z) = 0$. But

$$\underset{\substack{\Delta x \to 0\\ \Delta y \to 0}}{L} \sum\sum (z_1 - z_2)\Delta y\,\Delta x = \int_{x_1}^{x_2}\int_{y_1}^{y_2} (z_2 - z_1)\,dy\,dx$$
$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dz\,dy\,dx, \qquad (1)$$

where y_1 and y_2, in general functions of x, are the limiting values of y in the slice ST, and x_1 and x_2 are the limiting values of x so as to include all possible slices.

The *element of volume in the triple integral* is $dV = dx\,dy\,dz$.

The order in which the sum of parallelopipeds into columns and then into slices, and finally into the complete solid is made, may be varied. Thus, for example, we might have first summed for y keeping x and z fixed, obtaining horizontal rows of parallelopipeds parallel to the y-axis; then, keeping x fixed and summing for z, obtain vertical slices, or keeping z fixed and summing for x, obtain horizontal slices; the sum of the slices in either case gives the required volume. The integrals involved in these modes of procedure are

$$\int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} dy\,dz\,dx \quad \text{and} \quad \int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dy\,dx\,dz \qquad (2)$$

respectively. While the same symbols are used for the limits of the several variables in the various triple integrals giving the volume of a solid, it is understood that the values of these limits depend, in general, upon the order of integration. Since the limits for the variables are simpler for certain orders of integration than for others, advantage should be taken of the fact that there is a possible choice of order of integration.

In determining the limits of integration, it is necessary to have a clear picture of the bounding surface. While no general rule can be given, the following one will be found to work often. To fix the ideas, suppose that the order of the variables with respect to which the integration is effected is z, y, x. The limits for z are obtained by solving $F(x, y, z) = 0$ for z. Frequently those for y are obtained when $F(x, y, z_0) = 0$ is solved for y, where $z = z_0$ determines the largest section of the solid by a plane parallel to the xy-plane, when such a plane exists. Finally, the limits for x are the extreme values of this variable in this largest section.

All that has gone before applies equally well in case the solid is bounded by two or more surfaces; and also in case the surface or surfaces are not completely convex. The modifications will suggest themselves in any particular case.

Attention should be called to the fact that the actual work involved in calculating, as a triple integral, the volume enclosed by one or more surfaces is identical with that required when the volume is given as a double integral (§ 123). Moreover, whenever the areas of sections of the solid by planes parallel to one of the coördinate planes are known, the required volume can be obtained by evaluating a simple integral (§ 98).

EXERCISES

When practicable, find the following volumes by means of simple or double integrals:

1. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

2. Find the volume under the paraboloid $x^2 + y^2 = az$ cut out by the cylinder $x^2 + y^2 = 2\,ax$.

3. Find the volume cut from the cone $x^2 + y^2 - z^2 = 0$ by the cylinder $x^2 + y^2 - 2\,ax = 0$.

4. Find the volume cut from the paraboloid

$$x = 1 - \frac{y^2}{9} - \frac{z^2}{4}$$

by the yz-plane.

5. Find the volume cut off by the plane $y + z = a$ from the paraboloid $az = a^2 - x^2 - y^2$.

6. Find the volume in the first octant between the cylinders $z = 1 - x^2$ and $x = 1 - y^2$.

7. Find the volume in the first octant inside the cylinder $x^2 + y^2 = 2\,ax$ and outside the paraboloid $x^2 + y^2 = az$.

8. Find the volume of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$ cut off by the plane $z = c$.

9. Find the volume of $x^2 + y^2 = 2\,z$ cut off by $z = x + 1$.

10. Find the volume under $y^2 = a^2 - az$ cut out by the cylinder $x^2 + y^2 = a^2$.

129. Triple integral as the limit of a sum. Analogous to what was done in § 122, it will suffice here to present the facts in outline.

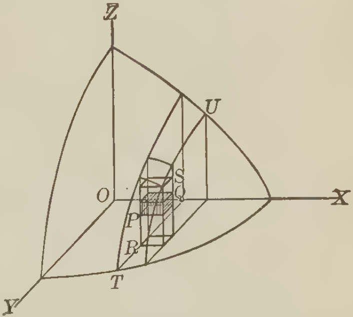

Fig. 119

Let $f(x, y, z)$ be a continuous function of x, y, z throughout R, a region of space bounded by one or more surfaces. As in § 128 we shall suppose this region divided into complete and partial parallelopipeds by planes parallel to the coördinate planes. If, as before, the coördinates of one of the vertices of the parallelopiped PQ (Fig. 119) are (x, y, z), consider the expression

$$f(x, y, z)\,\Delta x\,\Delta y\,\Delta z, \tag{1}$$

which is the product of the value of $f(x, y, z)$ at any point of PQ by the volume of the latter. Take the sum of these products, for all the parallelopipeds belonging to R,

$$\sum f(x, y, z)\,\Delta x\,\Delta y\,\Delta z. \tag{2}$$

It can be proved that *as* Δx, Δy, *and* Δz *approach zero, the sum* (2) *approaches a definite limit, which is designated by*

$$\iiint_{(R)} f(x, y, z)\,dx\,dy\,dz, \tag{3}$$

and is known as the triple integral of the function $f(x, y, z)$ *taken over the region* R. It can be proved also that the value of the triple integral (3) is equal to that of each of the six triple integrals

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dz\,dy\,dx,$$

$$\int_{x}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x, y, z)\,dy\,dz\,dx, \cdots, \tag{4}$$

the remaining ones arising by selecting the other possible orders of integration. As mentioned in § 128 attention should be called to the fact that, while the same symbols are used for the limits of the several variables in the integrals (4), it is understood that the values of these limits depend, in general, upon the order of integration.

Moreover, if the region R is broken up into elementary volumes ΔV, other than parallelopipeds, according to *any* law, provided the greatest dimensions of each of the elementary volumes approaches zero as their number grows indefinitely, it can be proved that the limit of the sum $\sum f(x, y, z)\Delta V$ approaches the same limit (3); *i.e.*

$$\iiint_{(R)} f(x, y, z)\,dV = \iiint_{(R)} f(x, y, z)\,dx\,dy\,dz. \tag{5}$$

In evaluating the triple integral (3), advantage should be taken of the fact that the ease or difficulty of determining the limits of integration, as well as that of the actual problem of integration itself, is often dependent upon the particular order of integration selected. As will be seen in the applications which follow in the succeeding sections of this chapter, one of the practical difficulties arising is the finding of the limits for the variable in the definite integrals.

Another reason for selecting the order of integration carefully is that often the result of the first partial integration, or even that of the first two partial integrations, may be known from geometrical or other considerations for a possible order of integration. In such a case the problem of evaluating the triple integral may be reduced to that of evaluating a double or simple integral respectively.

130. Volume in cylindrical coördinates. The element of volume suggested by the use of cylindrical coördinates (§ 125) is obtained if R, the region of space under consideration, is divided into smaller regions by first constructing a set of cylinders with the z-axis for axis and radii differing successively by the same amount $\Delta\rho$; then passing a set of planes through the axis, each making an angle $\Delta\theta$ with each of its neighbors; and finally passing a set of planes parallel to the xy-plane, Δz apart. The volume of such a small region is approximately $\rho\,\Delta\rho\,\Delta\theta\,\Delta z$; for it is approximately a rectangular parallelopiped whose dimensions are $PQ = \rho\,\Delta\theta$, $QR = \Delta\rho$, and $RS = \Delta z$ (Fig. 120). Taking the sum of these throughout R and passing to the

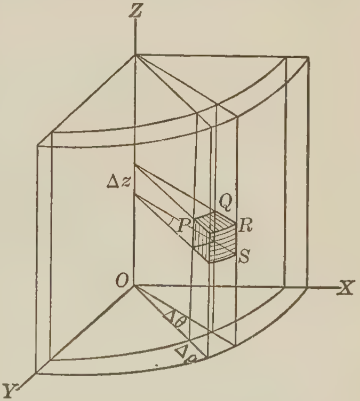

FIG. 120

limit as $\Delta\rho$, $\Delta\theta$, Δz approach zero, it can be shown that the volume of R is equal to

$$\underset{\substack{\Delta\rho\to 0\\ \Delta\theta\to 0\\ \Delta z\to 0}}{L}\sum\rho\,\Delta\rho\,\Delta\theta\,\Delta z=\iiint_{(R)}\rho\,d\rho\,d\theta\,dz.$$

The *element of volume in cylindrical coördinates in the triple integral* is $dV=\rho\,d\rho\,d\theta\,dz$.

If we integrate first for z, we have

$$V=\iint_{(S)}(z_2-z_1)\rho\,d\theta\,d\rho,$$

which is the same result as found in § 125. But the advantage of having the volume given as a triple integral is that we have the selection of the order of integration at our disposal.

131. Volume in spherical or space polar coördinates. The spherical coördinates of a point P may be taken as

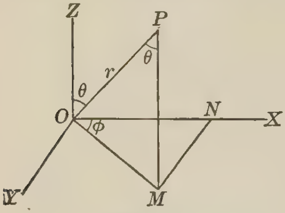

FIG. 121

$r\equiv OP$, the distance of P from the origin.

$\theta\equiv\angle POZ$, the angle which OP makes with the z-axis.

$\phi\equiv\angle MOX$, the angle which OM, the projection of OP on the xy-plane, makes with the x-axis.

The rectangular coördinates are given in terms of these by the formulas

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta.$$

Evidently

$r=$ const. is the equation of a sphere with center at O and radius r.

$\theta=$ const. is the equation of a cone with O as vertex, OZ as axis, and its vertical angle equal to 2θ.

$\phi=$ const. is the equation of a plane through OZ, making an angle ϕ with OX.

The element of volume, suggested by the use of spherical coördinates, is obtained if R, the region of space under consideration, is divided into smaller regions by constructing first a set of spheres with their centers at the origin and their radii differing successively by the same amount Δr; then constructing a set of cones with O as vertex, OZ as axis, with their vertical angles differing successively by $2\,\Delta\theta$; and finally, passing a set of planes through the z-axis, each making an angle $\Delta\phi$ with each of its neighbors. The volume of such a small region is, approximately, a rectangular parallelopiped whose dimensions are $PQ = r\,\Delta\theta$, $QR = r \sin\theta\,\Delta\phi$, and $RS = \Delta r$. Hence the volume of the small region is approximately $r^2 \sin\theta\,\Delta r\,\Delta\theta\,\Delta\phi$.

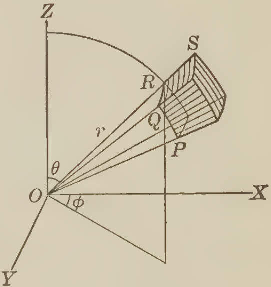

FIG. 122

Taking the sum of these throughout and passing to the limit as Δr, $\Delta\theta$, and $\Delta\phi$ approach zero, it can be shown that the volume of R is equal to

$$\underset{\substack{\Delta\gamma \to 0 \\ \Delta\theta \to 0 \\ \Delta\phi \to 0}}{L} \sum r^2 \sin\theta\,\Delta r\,\Delta\theta\,\Delta\phi = \iiint\limits_{(R)} r^2 \sin\theta\,dr\,d\theta\,d\phi.$$

The *element of volume in spherical coördinates in the triple integral* is $dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$.

Here again, in a given problem, advantage should be taken of the fact that the order of integration is optional. Thus for a sphere of radius a

$$V = 8\int_0^a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2 \sin\theta\,d\theta\,d\phi\,dr$$

$$= 8\int_0^a \int_0^{\frac{\pi}{2}} r^2\,d\phi\,dr = 4\,\pi\int_0^a r^2\,dr = \tfrac{4}{3}\,\pi a^3.$$

Stopping with the second partial integration, we see we may take $4\,\pi r^2\,dr$ as the *element of volume.* It is approximately the volume of a spherical shell of thickness dr and inner radius r. It is the *differential of volume of the sphere* when r is taken as the variable of reference. This will explain the rather striking fact that the derivative of the volume of a sphere $\frac{4}{3}\pi r^3$ equals $4\,\pi r^2$, the area of the surface of the sphere.

132. Mass; mean density. If m is the mass of a body and V its volume, $\dfrac{m}{V}$ is defined as the *mean density* of the body. Fixing our attention upon P, some point of the body, let us take a small part of the volume ΔV, containing P. If Δm is its mass, its mean density is $\dfrac{\Delta m}{\Delta V}$. Allowing ΔV to approach zero in such a way that each of its dimensions approaches zero, but at each stage it still contains P, the limit of the mean density, which we shall denote by $\dfrac{dm}{dV}$, is defined as the *density of the body at P.* Calling it δ, we have

$$\delta = \frac{dm}{dV}. \tag{1}$$

If δ is the same at all points of the body, the latter is said to be *homogeneous,* and δ is said to be the density of the body. The density of a homogeneous body is readily seen to be the mass of unit volume of the body.

Frequently, the density of a body at a point of it is a function of the coördinates of the point. By breaking up the body into smaller and smaller parts ΔV, according to some scheme suggested by the nature of the body or the way in which δ varies from point to point, we have the approximate value of the mass of each part ΔV given by

$$\Delta m = \delta\,\Delta V,$$

where δ is the density at some point of ΔV. Taking the sum, and passing to the limit as ΔV approaches zero, it can be shown that

$$m = \int \delta \, dV, \tag{2}$$

where the integral is to be taken over the entire body. The *element of mass* is $dm = \delta \, dV$, which is consistent with (1).

The integral in (2) may be a simple, double, or triple one, depending upon the element dV, which, in turn, is determined by the nature of the problem. As a rule, it is desirable to so select it as to involve as few integrations as possible.

Denoting the mean density by $\bar{\delta}$, we have from (2)

$$\bar{\delta} = \frac{\int \delta \, dV}{V}.$$

Ex. 1. Find the mass and mean density of a circular plate of radius a if the density at a point is proportional to the square of the distance of the point from the center.

Using polar coördinates $\delta = k\rho^2$, where k is a constant. Let the thickness of the plate be c. A convenient element of volume is a ring of radius ρ and breadth $d\rho$. Then $dV = 2\, c\pi\rho \, d\rho$.

$$\therefore m = 2\, ck\pi \int_0^a \rho^3 \, d\rho = \frac{ck\pi a^4}{2}.$$

Evidently

$$V = c\pi a^2. \quad \therefore \bar{\delta} = \frac{ka^2}{2}.$$

Ex. 2. Find the mass and mean density of a hemisphere of radius a if the density at a point varies as the distance of the latter from the base of the hemisphere.

Here $\delta = kz$. A convenient element of volume is a horizontal slice of thickness dz, the lower end being at a distance z from the base. The radius of this lower end of the slice is $\sqrt{a^2 - z^2}$. Hence $dV = \pi(a^2 - z^2)dz$.

$$\therefore m = k\pi \int_0^a z(a^2 - z^2)dz = \frac{k\pi a^4}{4}.$$

Moreover

$$V = \frac{2\pi a^3}{3}. \quad \therefore \ \bar{\delta} = \frac{3ka}{8}.$$

Ex. 3. Find the mass and mean density of a plate bounded by the parabola $y^2 = 4ax$, the x-axis and the latus rectum of the parabola, the density at any point of the plate being proportional to the product xy.

Here $\delta = kxy$. There is no special choice of dV available here. So we take $dV = c\,dx\,dy$, where c is the thickness of the plate.

$$\therefore \ m = ck\int_0^a\int_0^{2\sqrt{ax}} xy\,dy\,dx = \frac{2cka^4}{3}.$$

$$V = 2c\sqrt{a}\int_0^a \sqrt{x}\,dx = \frac{4ca^2}{3}. \quad \therefore \ \bar{\delta} = \frac{ka^2}{2}.$$

EXERCISES

Whenever possible select the element of volume so as to simplify the resulting integral to be evaluated.

1. Find the mass and mean density of a straight wire of length a if the density varies as the distance from one end.

2. Find the mass and mean density of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if the density varies (a) as the product xy; (b) as the sum $x + y$.

3. Find the mass and mean density of a circular plate of radius a whose density at a point varies (a) as the distance of the point from a given diameter; (b) as the square of the distance from a given diameter; (c) as the distance from the center.

4. Find the mass and mean density of a sphere of radius a whose density at a point varies (a) as the distance of the point from the center; (b) as the square of the distance from the center; (c) as the distance from a diameter.

5. Find the mass and mean density of a right circular cone of altitude a and base radius b, whose density at a point varies (a) as the distance of the point from the base; (b) as the distance from the axis of the cone; (c) as the distance from the vertex of the cone.

133. Moment. If the mass m of a very small particle is supposed concentrated at a single point, at a distance r from a given line or plane, the product rm is defined as the *moment of the particle* with respect to the line or plane.

If we have n such particles $m_1, m_2, \cdots m_n$, supposed concentrated at n points distant $r_1, r_2, \cdots, r_n$ respectively from the given line or plane, the sum of the products

$$\sum_{i=1}^{n} r_i m_i = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n$$

is defined as the *moment of the set of particles* with respect to the line or plane.

If we have a solid body (a continuum of particles), suppose it divided into small volumes ΔV of mass Δm. If the distance of some point of ΔV from the given line or plane is r, the limit of the sum $\sum r\,\Delta m$, as each of the dimensions of ΔV approaches zero, is defined as the *moment of the body* with respect to the line or plane. But this limit is a definite integral, either a simple, double, or triple one, depending upon the element ΔV. Hence we have

$$\textit{Moment of body} = \int r\,dm = \int r\,\delta\,dV.$$

If the body is homogeneous, δ is a constant and may be taken outside the integral sign. Frequently it is customary to consider $\delta = 1$ and speak of the moment of a geometrical figure, such as a curve, a surface, or a solid. For such a figure

$$\textit{Moment of figure} = \int r\,dV,$$

where V is used for content of the figure; so that dV may be ds, a length, or dA, an area.

In actual practice, it is often possible to reduce the number of integrations by a proper choice of element of volume or mass. Illustrative exercises and experience will serve to acquaint the student with such possibilities. In general we may say that this can be effected when it is possible to replace the aggregate of a set of elementary volumes or masses, each of which has the same moment, by a single new element.

Ex. 1. Find the moment of the area of a semicircle about its diameter.

Let the equation of the circle be $x^2 + y^2 = a$, and let the diameter, about which the moment is to be found, be the y-axis. This will evidently be twice the moment of the quarter-circle lying in the first quadrant. Thus we have, designating the moment by M_y,

$$M_y = 2\int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx.$$

The first partial integration with respect to y gives

$$M_y = 2\int_0^a x\sqrt{a^2 - x^2}\,dx.$$

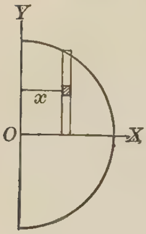

FIG. 123

We might have selected at the outset $dV = \sqrt{a^2 - x^2}\,dx = y\,dx$ as our element of volume, for every elementary portion $dx\,dy$ of the strip between the two ordinates at x and $x + dx$ is approximately at the same distance x from the line of reference.

Integrating again, we have

$$M_y = \frac{2\,a^3}{3}.$$

Ex. 2. Find the moment of a semi-circumference about its diameter.

Here
$$dV = ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \frac{a\,dx}{y}.$$

$$\therefore\ M_y = 2\,a\int_0^a \frac{x\,dx}{y} = 2\,a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} = 2\,a^2.$$

It is even simpler if we take y as the independent variable. Then $ds = \frac{a\,dy}{x}$, and

$$M_y = 2\int x\,ds = 2\,a\int_0^a dy = 2\,a^2.$$

Ex. 3. Find the moment of a cone about its base, if the density at each point is proportional to its distance from the base. Let a be the altitude of the cone and b the radius of its base.

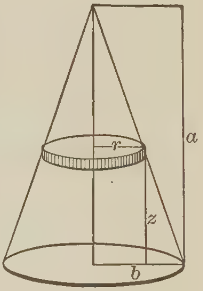

FIG. 124

Instead of using a triple integral in this case, we can take for dV a slice of the cone by two horizontal planes, at distances z and $z+dz$ from the base. If r is the radius of the lower section, we have by similar triangles

$$r = \frac{b(a-z)}{a}. \quad \therefore\ dV = \frac{\pi b^2(a-z)^2dz}{a^2}.$$

Moreover

$$\delta = kz. \quad \therefore\ dm = \frac{k\pi\, b^2z(a-z)^2dz}{a^2}.$$

$$\therefore\ \text{Moment} = \frac{k\,\pi b^2}{a^2}\int_0^a z^2(a^2-z)^2\,dz = \frac{k\pi\,a^3b^2}{30}.$$

Exercises involving the finding of moments will be found in the succeeding sections.

134. Center of gravity; of mass. Centroid. It can be proved that associated with a body there exists a point P such that if the entire mass of the body were concentrated at P, its moment with respect to any line or plane would be the same as for the body in its original condition. This point is known as the *center of gravity* or *center of mass* of the body. It is also frequently referred to as the *centroid*. Following the usual custom, we shall employ the term "center of gravity" only. As in § 133, when dealing with a geometrical figure, we shall assume the latter to be homogeneous, with $\delta = 1$.

It should be noted that the center of gravity is a unique point, and does not in any way depend upon the choice of coördinates.

The following theorems, useful in practice, result readily from the definition of the center of gravity. Their proofs will be left as exercises for the student.

Theorem I. *If* $(\bar{x}_1, \bar{y}_1, \bar{z}_1), (\bar{x}_2, \bar{y}_2, \bar{z}_2), \cdots, (x_n, \bar{y}_n, \bar{z}_n)$ *are the centers of gravity of* n *bodies whose masses are* $m_1, m_2, \cdots, m_n$ *respectively, the center of gravity of the aggregate is the point*

$$\bar{x} = \frac{\bar{x}_1 m_1 + \bar{x}_2 m_2 + \cdots + \bar{x}_n m_n}{m_1 + m_2 + \cdots + m_n}, \quad \bar{y} = \frac{\bar{y}_1 m_1 + \bar{y}_2 m_2 + \cdots + \bar{y}_n m_n}{m_1 + m_2 + \cdots + m_n},$$

$$\bar{z} = \frac{\bar{z}_1 m_1 + \bar{z}_2 m_2 + \cdots + \bar{z}_n m_n}{m_1 + m_2 + \cdots + m_n}.$$

Theorem II. *The moment of a body with respect to a line or plane through its center of gravity is zero.*

Theorem III. *The center of gravity of a body lies on a line or plane of symmetry of the body, if there is one.*

Attention should be called to the fact that the position of the center of gravity of a body, as well as that of a possible line or plane of symmetry, depends upon the distribution of mass in the body.

135. Center of gravity in the plane. The center of gravity of a plane figure is known when we have its coördinates, in particular its rectangular coördinates, usually designated by $(\bar{x}, \bar{y})$. If we denote the moments of the figure with respect to the x-axis and the y-axis by M_x and M_y respectively, we have by the definition of center of gravity

$$V\bar{x} = M_y \text{ and } V\bar{y} = M_x,$$

where V, as before, is used for the content of the figure. If the latter is an area bounded by one or more curves, V should be replaced by A; if it is the arc of a curve, V should be replaced by s, its length. Then

$$\bar{x} = \frac{M_y}{V} \text{ and } \bar{y} = \frac{M_x}{V},$$

where M_y and M_x are calculated by the methods of § 133.

Ex. 1. Find the center of gravity of the area enclosed by two parabolas $y^2 = ax$ and $x^2 = by$.

The curves intersect in the points (0, 0) and $(a^{\frac{1}{3}}b^{\frac{2}{3}}, a^{\frac{2}{3}}b^{\frac{1}{3}})$.

For $\bar{x}$, a convenient element of area is a strip between two ordinates at distances x and $x + dx$ from the y-axis. Then

$$\bar{x} = \frac{\displaystyle\int_0^{a^{\frac{1}{3}}b^{\frac{2}{3}}} x\left(\sqrt{ax} - \frac{x^2}{b}\right)dx}{\displaystyle\int_0^{a^{\frac{1}{3}}b^{\frac{2}{3}}} \left(\sqrt{ax} - \frac{x^2}{b}\right)dx} = \frac{9\,a^{\frac{1}{3}}b^{\frac{2}{3}}}{20}.$$

Similarly for $\bar{y}$, a convenient element of area is a strip between two horizontal lines at distances y and $y + dy$ from the x-axis. Then

$$\bar{y} = \frac{\displaystyle\int_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} y\left(\sqrt{by} - \frac{y^2}{a}\right)dy}{\displaystyle\int_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} \left(\sqrt{by} - \frac{y^2}{a}\right)dy} = \frac{9\,a^{\frac{2}{3}}b^{\frac{1}{3}}}{20}.$$

Ex. 2. Find the center of gravity of the area of one loop of the lemniscate $\rho^2 = a^2 \cos 2\,\theta$.

Here $dV = dA = \rho\, d\rho\, d\theta$ and $x = \rho \cos \theta$.

$$\therefore\ M_y = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \rho^2 \cos\theta\, d\rho\, d\theta = \frac{a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta(\cos 2\,\theta)^{\frac{3}{2}} d\theta$$

$$= \frac{a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta(1 - 2\sin^2\theta)^{\frac{3}{2}} d\theta.$$

Introducing a new variable by means of the transformation $\sqrt{2}\sin\theta = \sin u$, and noting that the value of the resulting definite integral is equal to twice the integral in which the limits are 0 and $\frac{\pi}{2}$, we have

$$M_y = \frac{a\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \cos^4 u\, du = \frac{a^3\pi\sqrt{2}}{16}.$$

Moreover

$$A = \tfrac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \rho^2\, d\theta = a^2\int_0^{\frac{\pi}{4}} \cos 2\,\theta\, d\theta = \frac{a^2}{2}.$$

$$\therefore\ \bar{x} = \frac{M_y}{A} = \frac{\pi a\sqrt{2}}{8}.$$

From symmetry it is evident that

$$\bar{y} = 0.$$

Ex. 3. Find the center of gravity of an arc of the cycloid

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$$

From symmetry it is evident that $\bar{x} = a\pi$.

To find y, we note that $dV = ds = 2\,a\sin\frac{\theta}{2}d\theta$.

$$\therefore\ M_x = \int y\, ds = 2\,a^2\int_0^{2\pi}(1 - \cos\theta)\sin\frac{\theta}{2}d\theta$$

$$= 4\,a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}d\theta = \frac{32\,a^2}{3}.$$

Moreover

$$s = 2\,a\int_0^{2\pi}\sin\frac{\theta}{2}d\theta = 8\,a.$$

$$\therefore\ \bar{y} = \frac{M_x}{s} = \frac{4\,a}{3}.$$

EXERCISES

Find the center of gravity of each of the following:

1. A straight line of length a.
2. An isosceles triangle of altitude a.

3. The triangle bounded by $y = mx$, $x = a$, $y = 0$.

4 Any triangle.

5. A quadrant of the area of the circle $x^2 + y^2 = a^2$.

6. A quadrant of the circumference of the circle $x^2 + y^2 = a^2$.

7. A quadrant of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

8. The joint area of the circles $x^2 + y^2 = a^2$ and $(x - b)^2 + y^2 = c^2$, where $b - c > a$. [Use Theorem I of § 134.]

9. The area remaining when the circle $x^2 + y^2 - ax = 0$ is cut out from the circle $x^2 + y^2 = a^2$.

10. The area remaining when one corner of a square of side a is cut off by a line joining the middle points of two adjacent sides.

11. The area remaining when there is removed from the quadrant of a circle of radius a, the triangle formed by the bounding radii and the line through the middle points of the latter.

12. The area of the parabola $y = 2x - x^2$ cut off by the x-axis.

13. The area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes.

14. The area of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant.

15. The arc of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant.

16. The area bounded by $y^2 = 4ax$, $x = b$, $y = 0$.

17. The area bounded by $y^2 = 4ax$, $x = 0$, $y = b$.

18. The area bounded by $y^2 = 4ax$ and $y = mx$.

19. The area under an arch of the cycloid.

20. The area under one arch of the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

21. The arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ between $x = 0$ and $x = a$.

22. The area under the catenary from $x = 0$ to $x = a$.

23. The area between the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote $x = 2a$.

24. The area of the upper half of the cardioid

$$\rho = a(1 + \cos\theta).$$

25. The arc of the upper half of the cardioid

$$\rho = a(1 + \cos\theta).$$

26. The area of one loop of $\rho = a \sin 2\theta$.

27. The area of the sector of a circle of angle 2α. The radius of the circle is a. [Use polar coördinates, taking the bisector of the angle for initial line.]

28. The arc of a circle of radius a subtending an angle 2α at the center.

136. Theorems of Pappus. In § 96 we saw that the volume generated by revolving the area bounded by a convex closed curve about the x-axis is given by the formula

$$V = \pi \int_{a_1}^{a_2} (y_2^2 - y_1^2)dx, \tag{1}$$

where y_2 and y_1 are the ordinates of the boundary corresponding to a given value of x. And this formula can be extended readily to the case where the boundary is not entirely convex. Evidently the volume is equal to that of the double integral

$$V = 2\pi \int_{x_1}^{x_2} \int_{y_1}^{y_2} y\,dy\,dx = 2\pi \iint_{(A)} y\,dA. \tag{2}$$

We have just seen (§ 135) that the y-coördinate of the center of gravity of the area is

$$\bar{y} = \frac{\displaystyle\iint_{(A)} y\,dA}{A}. \quad \therefore \iint_{(A)} y\,dA = \bar{y}A.$$

Hence

$$V = 2\pi \bar{y} A.$$

Expressed in words this is

Pappus' Theorem I. *If a plane area is revolved about an axis in its plane, which does not cross the area,*[1] *the volume generated is equal to the product of the area and the length of the path described by its center of gravity.*

In § 97 we saw that the surface generated by a curve revolved about the x-axis is

$$S = 2\pi \int_{(s)} y\, ds,$$

where the integral is taken along the entire curve. But we have just seen in § 135 that the y-coördinate of the center of gravity of the curve is

$$\bar{y} = \frac{\int_{(s)} y\, ds}{s}. \quad \therefore \int_{(s)} y\, ds = \bar{y} s.$$

Hence

$$S = 2\pi \bar{y} s.$$

Expressed in words this is

Pappus' Theorem II. *If an arc of a plane curve is revolved about an axis in its plane, which does not cross the arc,*[2] *the area generated is equal to the product of the length of arc and the length of the path described by its center of gravity.*

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[1] Algebraically the theorem is still true when the axis crosses the revolving area. But in this case the parts of the volume generated by the portions of the area on the two sides of the axis must be taken with opposite algebraic signs. For y in (2) is positive on one side of the x-axis and negative on the other.

[2] Here also the theorem is still true, when the axis crosses the revolving arc, provided part of the generated area is taken as negative.

EXERCISES

1. Find the volume and lateral area of a right circular cone, making use of the fact that such a cone is generated when a right triangle is revolved about one of its perpendicular sides.

2. Making use of Pappus' Theorems and the known values of volume and surface of a sphere, find the center of gravity of the area and of the arc of a semicircle.

3. Find the volume and area of the solid (a *torus*) generated by revolving a circle of radius a about an axis in its plane at a distance b from the center of the circle. [Take $b>a$.]

4. Find the volume generated by revolving the half of an ellipse above the major axis about an axis parallel to the major axis and at a distance c below it. Let the major and minor axes be $2a$ and $2b$ respectively.

137. Center of gravity in space. If the moments of the mass of a body with respect to the coördinate planes are M_{yz}, M_{xz}, M_{xy}, respectively, and if $\bar{x}$, $\bar{y}$, $\bar{z}$ are coördinates of the center of gravity,

$$m\bar{x} = M_{yz},\quad m\bar{y} = M_{xz},\quad m\bar{z} = M_{xy},$$

where m is the mass of the body. Then

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\int x\,dm}{m},\quad \bar{y} = \frac{M_{xz}}{m} = \frac{\int y\,dm}{m},\quad \bar{z} = \frac{M_{xy}}{m} = \frac{\int z\,dm}{m}.$$

If the body is homogeneous, its density may be taken equal to unity in finding the coördinates of the center of gravity, since the latter are independent of the value of the density in this case. For if it has any other value δ,

$$dm = \delta\,dV \quad\text{and}\quad m = \delta V.$$

$$\therefore\ \bar{x} = \frac{\int x\,dm}{m} = \frac{\delta\int x\,dV}{\delta V} = \frac{\int x\,dV}{V};$$

and similarly for $\bar{y}$ and $\bar{z}$.

Ex. 1. Find the center of gravity of a homogeneous cylinder.

Let the altitude be a and the area of the base B. Any section parallel to the base is also b in area. If the plane of the base is taken as the yz-plane, and if for element of volume is selected a slice between two sections of the cylinder by planes parallel to the base and at distances x and $x + dx$, respectively, from it, $dV = Bdx$ and

$$M_{yz} = \int x\,dV = B\int_0^a x\,dx = \frac{Ba^2}{2}.$$

$$V = Ba. \quad \therefore\ \bar{x} = \frac{a}{2}.$$

The line joining the centers of gravity of the two bases is known as the *axis of the cylinder. The center of gravity of the cylinder lies on the axis midway between the bases.* The cylinder may be either oblique or right.

Ex. 2. Find the center of gravity of a homogeneous cone or pyramid.

Let the altitude of the cone or pyramid be a and the area of the base B. If A is the area of a section of the solid parallel to the base at a distance x from the vertex, it is known that

$$\frac{A}{B} = \frac{x^2}{a^2}. \quad \therefore\ A = \frac{Bx^2}{a^2}.$$

If we select our coördinate axes so that the origin is at the vertex of the cone or pyramid and the yz-plane is parallel to its base, we find M_{yz} by taking for element of volume the slice between two planes parallel to the yz-plane at distances x and $x + dx$ from it. Then the element of volume is $dV = \dfrac{Bx^2\,dx}{a^2}$.

$$\therefore\ M_{yz} = \int x\,dV = \frac{B}{a^2}\int_0^a x^3\,dx = \frac{Ba^2}{4}.$$

Moreover,

$$V = \int dV = \frac{B}{a^2}\int_0^a x^2\,dx = \frac{Ba}{3},$$

as is well known. Hence

$$\bar{x} = \frac{3a}{4}.$$

For the other coördinates of the center of gravity, we note that since sections of the cone or pyramid parallel to the base are similar figures, the locus of their center of gravity is a straight line through the vertex. This line is known as the *axis of the cone or pyramid. The center of gravity lies on the axis, and is three quarters the distance along it from the vertex to the base.*

Ex. 3. Find the center of gravity of a right circular cone, whose density at a point is proportional to its distance from the axis.

Because of symmetry, the center of gravity lies on the axis of the cone. We shall take this axis as the z-axis of coördinates and the plane of the base of the cone as the xy-plane.

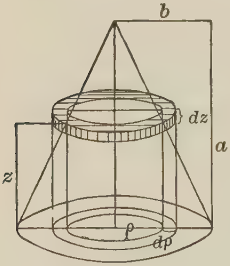

FIG. 125

In this case it will be desirable to use cylindrical coördinates. Then the density at any point (ρ, θ, z) is $\delta = k\rho$, where k is a constant. A desirable element of volume is $dV = 2\pi\rho\,d\rho\,dz$, the circular ring included between two horizontal planes at distances z and $z + dz$ above the xy-plane and two circular cylinders co-axial with the cone and having for radii of their directrices ρ and $\rho + d\rho$ respectively. (See Fig. 125.) Since, when $d\rho$ is small, each point of this elementary volume is approximately at the distance ρ from the axis, the element of mass is $dm = 2\pi k\rho^2\,d\rho\,dz.$

If the altitude of the cone is a and the radius of the base is b, we have by similar triangles $\dfrac{\rho}{a-z}=\dfrac{b}{a}$,

$$z=a-\frac{a\rho}{b} \text{ and } \rho=\frac{b}{a}(a-z).$$

$$\therefore M_{xy}=\int z\,dm=2\,\pi k\int_0^a\int_0^{\frac{b}{a}(a-z)} z\rho^2\,d\rho\,dz=\frac{\pi k\,b^3a^2}{30}.$$

To evaluate m integrate first with respect to ρ and then z.

Thus, $$m=\int dm=2\,\pi k\int_0^b\int_0^{a-\frac{a\rho}{b}} z\rho^2\,dz\,d\rho=\frac{\pi k\,b^3a}{6}.$$

[This suggests an element of mass that enables one to find the mass by means of a single integration. See Fig. 125.]

$$\therefore \bar{z}=\frac{M_{xy}}{m}=\frac{a}{5}.$$

Ex. 4. Find the center of gravity of a semicircular plate of radius a, if the density at any point is proportional to the square of its distance from center.

The plate may be considered either so thin as to enable one to ignore its third dimension, or one may consider it as a cylinder, in which case, the center of gravity lies midway between its bases. In either case we are concerned with a two-dimensional problem only.

Using polar coördinates to find the mass, we shall take for element of area $dA=\pi\rho\,d\rho$, cut out of the semicircle by two concentric circles of radii ρ and $\rho+d\rho$ respectively. Then $dm=\pi k\rho^3\,d\rho$.

$$\therefore m=\pi k\int_0^a \rho^3\,d\rho=\frac{\pi ka^4}{4}.$$

Still using the polar coördinates, to find the moments of mass with respect to the axes, we take for element of area, as in the general case, $dA = \rho\, d\rho\, d\theta$ and $dm = k\rho^3\, d\rho\, d\theta$. Suppose the diameter of the semicircle is taken as the y-axis.

$$M_y = \int x\, dm = k\int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^4 \cos\theta\, d\theta\, d\rho = \frac{2\,ka^5}{5}.$$

$$\therefore\ \bar{x} = \frac{M_y}{m} = \frac{8\,a}{5}.$$

Because of symmetry $M_x = 0$ and $\bar{y} = 0$.

EXERCISES

Find the center of gravity of each of the following:

1. An octant of a sphere of radius a.

2. A hemisphere whose density at a point is proportional to its distance from the plane face.

3. A hemisphere whose density at a point is proportional to its distance from the center.

4. The paraboloid of revolution formed by revolving about the x-axis the parabola $y^2 = 4\,ax$ from $x = 0$ to $x = b$.

5. A homogeneous solid, composed of a right circular cylinder of altitude a and base radius b surmounted by a right circular cone of altitude c and base radius b.

6. A rod of length a and uniform thickness whose density at a point is (a) proportional to its distance from one end; (b) proportional to the square of its distance from one end.

7. The solid remaining when a right circular cone of base radius b and altitude a is removed from a right circular cylinder having the same base and altitude.

8. The part of the conoid $x^2z^2 + a^2y^2 = b^2x^2$ between the planes $x = 0$ and $x = a$, (a) if the conoid is homogeneous; (b) if the density at a point is proportional to its distance from the yz-plane.

138. Moment of inertia. Radius of gyration. If the mass m of a very small particle is supposed concentrated at a single point at a distance r from a given line, the product

$$r^2 m \equiv I_l$$

is defined as the *moment of inertia of the particle about the line.*

Given a set of particles $m_1, m_2, \cdots, m_n$ supposed concentrated at n points distant $r_1, r_2, \cdots, r_n$, respectively, from the given line, the sum of the products

$$\sum_{i=1}^{n} r_i^2 m_i = r_1^2 m_1 + r_2^2 m_2 + \cdots + r_n^2 m_n \equiv I_l$$

is defined as the *moment of inertia of the set of particles about the line.*

In a manner entirely analagous to that employed in § 133 for the moment of a solid with respect to a line or plane we arrive at the definition.

$$\begin{aligned} I_l &\equiv \textit{moment of inertia of body with respect to a line} \\ &= \int r^2\,dm = \int r^2 \delta\,dV, \end{aligned}$$

where the integral involved is simple, double, or triple, depending upon the nature of the element of volume dV selected.

If the body is homogeneous, δ is a constant. It is customary in this case to let $\delta = 1$ and speak of

$$I_l = \int r^2 dV$$

as the *moment of inertia of the solid about the line.* Here dV is used in a general sense, and should be replaced by ds in the case of a curve, and by dA in the case of an area.

If I_l is the moment of inertia of a body with respect to a line l, and m is its mass, their ratio is defined as the square of the *radius of gyration with respect to l.* Thus

$$\frac{I_l}{m} = k^2, \tag{1}$$

k being the radius of gyration. Clearing of fractions

$$I_l = k^2 m. \tag{2}$$

From (2) we see that the radius of gyration of a body with respect to a line is a number k, such that the moment of inertia of the body with respect to the line is the same as it would be if its entire mass were concentrated at a point at a distance k from the line.

Ex. 1. Find the moment of inertia, and the square of the radius of gyration of the area of the parabola $y^2 = 4\,ax$ cut off by $x = a$ and lying above the x-axis, (a) about the x-axis; (b) about the y-axis.

We shall designate these moments of inertia by I_x and I_y respectively.

$$I_x = \int y^2\,dA = \int_0^{2a}\int_{\frac{y^2}{4a}}^{a} y^2\,dx\,dy = \int_0^{2a} y^2\left(a - \frac{y^2}{4\,a}\right)dy = \frac{16\,a^4}{15}.$$

Since each point of an elementary area of the parabola lying between the two horizontal lines at distances y and $y + dy$ from the x-axis is approximately at the same distance from the x-axis, we could have taken this strip for element of area, thus $dA = \left(a - \frac{y^2}{4\,a}\right)dy$. This would have made unnecessary the first integration.

$$A = \int^{2a}\left(a - \frac{y^2}{4\,a}\right)dy = \frac{4\,a^2}{3}. \quad \therefore\ k^2 = \frac{I_x}{A} = \frac{4\,a^2}{5}.$$

For I_y we shall save one integration by taking

$$dA = y\,dx = 2\sqrt{ax}\,dx.$$

Then

$$I_y = \int x^2\,dA = 2\sqrt{a}\int_0^a x^{\frac{5}{2}}\,dx = \frac{4\,a^4}{7}.$$

$$\therefore\ k^2 = \frac{I_y}{A} = \frac{3\,a^2}{7}.$$

Ex. 2. Find the moment of inertia and the square of the radius of gyration of either half of the cardioid $\rho = a(1 + \cos\theta)$ about the x-axis.

Here $y = \rho\sin\theta$ and $dA = \rho\,d\rho\,d\theta$.

$$\therefore\ I_x = \int y^2 dA = \int_0^{\pi}\int_0^{a(1+\cos\theta)} \rho^3\sin^2\theta\,d\rho\,d\theta = \frac{21\,\pi a^4}{64}.$$

$$A = \frac{a^2}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta = \frac{3\,\pi a^2}{4}.\quad \therefore\ k^2 = \frac{I_x}{A} = \frac{7\,a^2}{16}.$$

Ex. 3. Find the moment of inertia and the square of the radius of gyration about the x-axis of an arc of the cycloid

$$x = a(\theta - \sin\theta),\quad y = a(1 - \cos\theta),$$

from $\theta = 0$ to $\theta = 2\,\pi$.

Here $$ds = 2\,a\sin\frac{\theta}{2}\,d\theta.$$

$$\therefore\ I_x = \int y^2\,ds = 2\,a^3\int_0^{2\pi}(1-\cos\theta)^2\sin\frac{\theta}{2}\,d\theta$$

$$= 8\,a^3\int_0^{2\pi}\sin^5\frac{\theta}{2}\,d\theta = \frac{256\,a^3}{15}.$$

$$s = 8\,a.\quad \therefore\ k^2 = \frac{I_x}{s} = \frac{32\,a^2}{15}.$$

Ex. 4. Find the moment of inertia and the square of the radius of gyration about the z-axis of the part of the cylinder $x^2 + y^2 - 2\,ax = 0$ included between $z = 0$ and $z = h$, if the density at a point of the cylinder is proportional to its distance from the xy-plane.

In rectangular coördinates

$$\delta = cz, \quad dm = cz\,dx\,dy\,dz, \quad r^2 = x^2 + y^2.$$

$$\therefore I_z = \int r^2\,dm = c\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\int_0^h z(x^2+y^2)dz\,dy\,dx.$$

While this can be integrated, there will be marked simplification in using cylindrical coördinates. Then

$$\delta = cz, \quad dm = c\,z\rho\,d\rho\,d\theta\,dz, \quad r^2 = \rho^2, \quad \rho = 2\,a\cos\theta.$$

$$\therefore I_z = c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\int_0^h z\rho^3\,dz\,d\rho\,d\theta = \frac{3\,ch^2\,\pi a^4}{4}.$$

$$m = \frac{ch^2\,\pi a^2}{2}. \quad \therefore k^2 = \frac{3\,a^2}{2}.$$

139. Theorems concerning moments of inertia. The moment of inertia of a body about a line plays an important rôle in Mechanics. To determine it in the case of certain bodies, it is convenient to introduce the notion of moment of inertia of a body with respect to a plane, even if this is of no interest in itself. If r is the distance of the element of mass dm of the body from a given plane p, then

$$I_p = \int r^2\,dm = \int r^2\delta\,dV.$$

is defined as the *moment of inertia of the body with respect to the plane p.* If the body is homogeneous, it is customary, as before, to let $\delta = 1$, and to define

$$I_p = \int r^2\,dV$$

as the *moment of inertia of the figure with respect to the plane.*

THEOREM I. *The moment of inertia of a body about a line is equal to the sum of its moments of inertia with respect to two mutually perpendicular planes through the line.*

Taking the line as the z-axis and the mutually perpendicular planes through it as the xz- and yz-planes, we have

$$I_z = \int (x^2 + y^2)\, dm = \int x^2\, dm + \int y^2\, dm.$$

But $$\int x^2\, dm = I_{yz} \quad \text{and} \quad \int y^2\, dm = I_{xz}.$$

$$\therefore I_z = I_{yz} + I_{xz},$$

which proves the theorem.

Similarly it is convenient to speak of the moment of inertia of a plane area about a point P in its plane, or about a line through P perpendicular to the plane. If r is the distance of an element of area dA from P, the *moment of inertia of the area about P* is

$$I_P = \int r^2\, dA.$$

It is sometimes called the *polar moment.*

The moment of inertia, with respect to either a line or a point, of a thin plate or lamina is obtained from the corresponding moment of inertia of its surface by replacing dA by dm, the element of mass of the lamina.

Theorem II. *The moment of inertia of an area* (*or lamina*) *about a point in its plane is equal to the sum of its moments of inertia about two mutually perpendicular lines through the point and lying in the plane.*

Taking the point as the origin and the mutually perpendicular lines through it as the coördinate axes,

$$I_O = \int (x^2 + y^2)\, dA = \int x^2\, dA + \int y^2\, dA.$$

But $$\int x^2\, dA = I_y \quad \text{and} \quad \int y^2\, dA = I_x.$$

$$\therefore I_O = I_y + I_x,$$

which proves the theorem.

THEOREM III. *The moment of inertia of a body about a given line is equal to its moment of inertia about another line, parallel to the given one through the center of gravity of the body, plus the moment of inertia about the given line of a particle having the mass of the body and located at its center of gravity.*

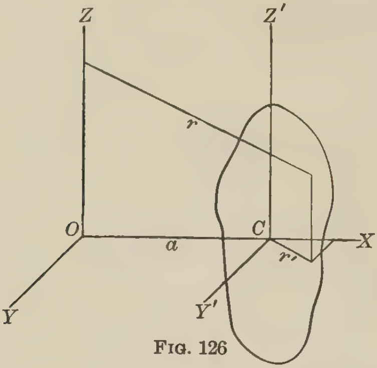

FIG. 126

Let the given line be the z-axis in Fig. 126, and let the center of gravity of the body lie on the x-axis at a distance a along it. The moment of inertia of the body about the z-axis is

$$I_z = \int r^2\, dm = \int (x^2 + y^2)\, dm.$$

Its moment of inertia about CZ', the line through the center of gravity parallel to the z-axis, is

$$I_c = \int r'^2\, dm = \int [(x-a)^2 + y^2]\, dm$$
$$= \int (x^2 + y^2)\, dm + a^2 \int dm - 2\,a \int x\, dm.$$

Since $C \equiv (a, 0, 0)$ is the center of gravity of the body

$$a = \frac{\int x\, dm}{m}. \quad \therefore \int x\, dm = am.$$

$$\therefore I_c = \int (x^2 + y^2) dm + a^2 m - 2\,a^2 m$$
$$= I_z - a^2 m.$$
$$\therefore I_z = I_c + a^2 m,$$

which proves the theorem.

Theorems entirely analogous to Theorem III hold for the (polar) moment of inertia of a plane figure about a point, and for the moment of a body about a plane. The statement and proof of these theorems will be left as exercises for the student.

THEOREM IV. *The radius of gyration of a homogeneous beam (or cylinder) about a line parallel to its elements is equal to the radius of gyration of a cross section about the same line.*

Taking the line as the z-axis, the moment of inertia of the beam about it is

$$I_z = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}(x^2+y^2)\,dz\,dy\,dx$$

$$= (z_2 - z_1)\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x^2+y^2)\,dy\,dx,$$

since the limits for z are constants.

But the moment of inertia of a cross section about the same line is

$$I_z' = \int_{x_1}^{x_2}\int_{y_1}^{y_2}(x^2+y^2)\,dy\,dx.$$

$$\therefore I_z = (z_2 - z_1)I_z'. \tag{1}$$

If A is the area of the cross section, the volume of the beam equals $(z_2 - z_1)A$. If k and k' are the respective radii of gyration of the beam and cross section, we have

$$I_z = k^2 V = k^2(z_2 - z_1)A.$$

$$I_z' = k'^2 A.$$

Using (1)

$$k^2(z_2 - z_1)A = (z_2 - z_1)k'^2 A. \quad \therefore k^2 = k'^2,$$

which proves the theorem.

THEOREM V. *If there are n bodies whose masses are* $m_1, m_2, \cdots, m_n$, *and their radii of gyration with respect to a line or plane are* $k_1, k_2, \cdots, k_n$, *respectively, the radius of gyration of the aggregate is equal to*

$$k^2 = \frac{k_1^2 m_1 + k_2^2 m_2 + \cdots + k_n^2 m_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} k_i^2 m_i}{\sum m_i}.$$

This follows at once from the definition of moment of inertia. It should be employed in finding the moments of inertia in case a single body can be broken up into simpler ones whose moments of inertia are known or can be found readily.

The theorems given in this section often enable one to simplify materially the problem of finding the moment of inertia of a given body. They should be thoroughly mastered and borne in mind when attacking a problem.

Ex. 1. Find the moment of inertia and the square of radius of gyration of a circle of radius a (*a*) about its center; (*b*) about a diameter; (*c*) about a tangent.

(*a*) Use polar coördinates, taking the center of the circle for origin. A convenient element of area is a circular ring concentric with the circle, and of breadth $d\rho$. Then $dA = 2\pi\rho\, d\rho$, and the moment of inertia about the origin is

$$I_O = \int \rho^2\, dA = 2\pi \int_0^a \rho^3\, d\rho = \frac{\pi a^4}{2}.$$

Since $$A = \pi a^2,\ k^2 = \frac{I_O}{A} = \frac{a^2}{2}.$$

(*b*) To find the moment of inertia about a diameter, we make use of the fact that because of symmetry the moment of inertia with respect to all diameters is the same. In particular, if I_x and I_y are the moments of inertia about two mutually perpendicular diameters, we have $I_x = I_y$ and (from Theorem II) $I_x + I_y = I_O$.

$$\therefore I_x = \tfrac{1}{2} I_O = \frac{\pi a^4}{4}.$$

Hence for a diameter, $k^2 = \dfrac{a^2}{4}$.

(*c*) To find I_t, the moment of inertia about a tangent, we make use of Theorem III. Then

$$I_t = I_x + a^2 A = \frac{\pi a^4}{4} + \pi a^4 = \frac{5\,\pi a^4}{4}.$$

Hence for a tangent, $k^2 = \dfrac{5\,a^2}{4}$.

Ex. 2. Find the moment of inertia and the square of the radius of gyration of a right circular cylinder of altitude h and base radius a, about a diameter of the base.

While this problem could be attacked directly, it will be much simpler to find the moments of inertia with respect to the plane of the base and the plane through the given diameter of the base and the axis of the cylinder, and then use Theorem I.

Using rectangular coördinates, let the axis of the cylinder be the z-axis, the plane of the base the xy-plane, and the given diameter the x-axis.

To find I_{xy}, the moment of inertia with respect to the xy-plane, take for element of volume a horizontal slice of thickness dz. Then $dV = \pi a^2\,dz$, and

$$I_{xy} = \int z^2\,dV = \pi a^2 \int_0^h z^2\,dz = \frac{\pi a^2 h^3}{3}.$$

To find I_{xz}, the moment of inertia with respect to the xz-plane, it will be simpler to first find I_z, the moment of inertia about the z-axis. Using cylindrical coördinates, take for element of volume a hollow cylinder, concentric with the given one and of thickness $d\rho$. Then $dV = 2\pi h\rho\, d\rho$, and

$$I_z = \int \rho^2\, dV = 2\pi h \int_0^a \rho^3\, d\rho = \frac{\pi a^4 h}{2}.$$

Incidentally we may note that in this case $k^2 = \frac{a^2}{2}$. Compare these results with those of Ex. 1 (*a*).

Noting that because of symmetry $I_{xz} = I_{yz}$, and, because of Theorem I, $I_{xz} + I_{yz} = I_z$, we have

$$I_{xz} = \tfrac{1}{2} I_z = \frac{\pi a^4 h}{4}.$$

Finally, because of Theorem I, the required moment of inertia is

$$I_x = I_{xy} + I_{xz} = \frac{\pi a^2 h^3}{3} + \frac{\pi a^4 h}{4} = \frac{\pi a^2 h}{12}(4h^2 + 3a^2).$$

Hence

$$k^2 = \frac{I_x}{V} = \frac{4h^2 + 3a^2}{12}.$$

Ex. 3. Find the moment of inertia and the square of the radius of gyration of a right circular cylinder about its axis, if its density at any point varies as its distance from the axis.

Let h be the altitude of the cylinder, and a the base radius. Use cylindrical coördinates with the axis of the cylinder as the z-axis. Then $\delta = c\rho$.

A convenient element of volume is that of a hollow cylinder coaxial with the given cylinder and of thickness $d\rho$. For this $dV = 2\pi h\rho\, d\rho$ and $dm = 2\pi ch\rho^2\, d\rho$.

$$\therefore I_z = \int \rho^2\, dm = 2\pi ch \int_0^a \rho^4\, d\rho = \frac{2\pi ch a^5}{5}.$$

$$m = \int dm = 2\pi ch \int_0^a \rho^2\, d\rho = \frac{2\pi ch a^3}{3}. \quad \therefore k^2 = \frac{3a^2}{5}.$$

It will be noted that I_z is proportional to h, the altitude, and k is independent of h and c. Thus the radius of gyration of this cylinder also about its axis is the same as that of a right section of it about the same line. Compare Theorem IV.

Ex. 4. Find the moment of inertia and the square of the radius of gyration of a right circular cylinder about its axis, if its density at any point varies as its distance from the base.

Let h be the altitude and a the base radius as before. Here $\delta = cz$. Using cylindrical coördinates, a convenient element of volume is a horizontal slice, of thickness dz, of a hollow cylinder, $d\rho$ in thickness (see Fig. 125, p. 432). Then $dV = 2\pi\rho\, d\rho\, dz$, and $dm = 2\pi cz\rho\, d\rho dz$. The moment of inertia about the z-axis is then

$$I_z = \int \rho^2\, dm = 2\pi c \int_0^h \int_0^a z\rho^3\, d\rho\, dz = \frac{\pi c h^2 a^4}{4}.$$

$$m = \int dm = 2\pi c \int_0^h \int_0^a z\rho\, d\rho\, dz = \frac{\pi c h^2 a^2}{2}. \quad \therefore k^2 = \frac{a^2}{2}.$$

If we ask for the moment of inertia of the cylinder with respect to an element of it, use may be made of Theorem III. Then

$$I_l = I_z + a^2 m = \frac{\pi c h^2 a^4}{4} + \frac{\pi c h^2 a^4}{2} = \frac{3\pi c h^2 a^4}{4}.$$

Compare illustrative Ex. 4 of § 138.

Ex. 5. Find the moment of inertia and the square of the radius of gyration about the center of the larger circle of radius a (Fig. 127) of the part remaining when the smaller circle of radius $\frac{a}{2}$ is removed.

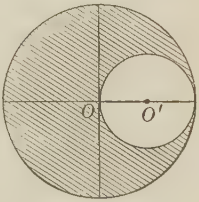

FIG. 127

In Ex. 1 we saw that the moment of inertia of the larger circle about O is

$$I_O^C = \frac{\pi a^4}{2}.$$

Using the same formula, the moment of inertia of the smaller circle about its center O' is

$$I^c_{O'} = \frac{\pi a^4}{32}.$$

To find I^c_O, the moment of inertia of the smaller circle about O, we make use of the analogue of Theorem III,

$$I^c_O = I^c_{O'} + \left(\frac{a}{2}\right)^2 \frac{\pi a^2}{4} = \frac{3\,\pi a^4}{32}.$$

The required moment of inertia is

$$I_O = I^C_O - I^c_O = \frac{\pi a^4}{2} - \frac{3\,\pi a^4}{32} = \frac{13\,\pi a^4}{32}.$$

To find the radius of gyration we note that the area of the figure is $\pi a^2 - \frac{\pi a^2}{4}$ or $\frac{3\,\pi a^2}{4}$.

$$\therefore k^2 = \frac{13\,a^2}{24}.$$

EXERCISES

Find the moment of inertia and the square of the radius of gyration of the following objects:

1. A rectangle of sides $2\,a$ and $2\,b$, (a) about a line through the center perpendicular to a side $2\,a$; (b) about the center.

2. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a$) about the major axis; (b) about the minor axis; (c) about the center.

3. A parallelopiped of edges $2\,a$, $2\,b$, and $2\,c$, (a) about the line through the center perpendicular to the plane containing edges $2\,a$ and $2\,b$; (b) about an edge $2\,c$.

4. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about the x-axis.

5. A right circular cone of altitude h and base radius a, (a) about its axis; (b) about a diameter of its base; (c) about a line through the vertex and perpendicular to the axis.

6. A ring bounded by two concentric circles of radii a_1 and a_2 respectively, (*a*) about the center; (*b*) about a diameter.

7. The area in illustrative Ex. 5 (Fig. 127), (*a*) about the common diameter OO'; (*b*) about O'; (*c*) about the diameter of the larger circle tangent to the smaller one.

8. A hollow right circular cylinder of altitude h and radii of circles of base a_1 and a_2, respectively, (*a*) about the axis of the cylinder; (*b*) an element of the outer surface; (*c*) a diameter of the base.

9. A beam of length h, about a line parallel to an edge of the beam and through the center of gravity, if the cross section is T-shaped as indicated in Fig. 128 with the dimensions given there.

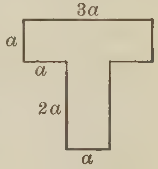

FIG. 128

10. A beam of length h, about a line parallel to an edge of the beam and through the center of gravity, if the cross-section is L-shaped as indicated in Fig. 129.

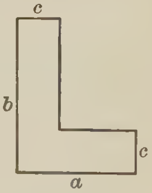

FIG. 129

11. An equilateral triangle about a median.

12. A square about a diagonal.

13. One loop of $\rho = a \sin 2\theta$, (*a*) about the origin; (*b*) about the x-axis.

14. The upper half of the cardioid $\rho = a(1 + \cos\theta)$ about the x-axis.

15. The lemniscate $\rho^2 = a^2 \cos 2\theta$, (*a*) about the y-axis; (*b*) about the origin.

16. The arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

17. The area of an arch of the cycloid about the x-axis.

18. An arc of a circle of radius a which subtends an angle 2α at the center, (*a*) about the center of the circle; (*b*) about its middle point; (*c*) about the diameter through its middle point; (*d*) about the tangent at its middle point.

19. A sector of angle 2α, of a circle of radius a, (a) about the center of the circle; (b) about the middle point of the bounding arc; (c) about the diameter through the middle point of the bounding arc; (d) about the tangent at the middle point of the bounding arc.

20. A straight wire of negligible diameter and of length h, about one end, if the density at any point is proportional to its distance from that end.

140. Attraction. Two particles of masses m_1 and m_2, respectively, supposed concentrated at two points separated by a distance r, attract each other with a force

$$\frac{km_1m_2}{r^2},$$

where k is a constant depending upon the choice of units of mass, of distance, and of force. Moreover this force is directed along the line joining the two particles. In particular, if one of the masses, say m_2, equals unity, we see that the force of attraction of m_1 upon a unit mass at a distance r is equal to

$$\frac{km_1}{r^2}$$

directed along the line joining the masses.

With this as a basis, let us ask what is the force of attraction of a given body of mass m upon a particle of unit mass located at a given point?

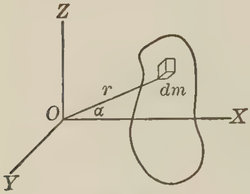

FIG. 130

Let the unit mass be located at the origin O in Fig. 130. The attraction of an elementary mass Δm of the body upon the unit mass is

$$\frac{k\,\Delta m}{r^2}.$$

Since the direction of the force of attraction of Δm varies with the position of Δm, we shall decompose each elementary attraction into its components in directions along the coördinate axes and sum these components. In particular let us find the component in the direction of the x-axis. If the line joining O with a point of Δm makes an angle α with the x-axis, the x-component of the attraction of Δm is

$$\frac{k \cos \alpha \, \Delta m}{r^2}.$$

The limit of the sum $\sum \frac{k \cos \alpha \, \Delta m}{r^2}$, as each of the dimensions of the elementary parts of the body approaches zero is defined as the *x-component of the force of attraction of the body.* Representing it by F_x and noting that the limit of the sum is a definite integral, simple, double, or triple, depending upon the form of the attracting body, and the choice of element of mass, we shall write

$$F_x = \int \frac{k \cos \alpha \, dm}{r^2}. \qquad (1)$$

Analogous formulas can be found for the components of the force of attraction in other directions.

The following examples will illustrate.

Ex. 1. Find the attraction of a homogeneous straight wire of length $2a$ and mass m upon a unit mass at a distance b from its middle point.

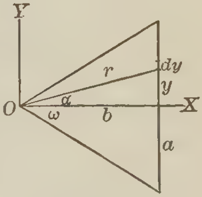

FIG. 131

Take the unit mass at the origin, and the x-axis perpendicular to the wire. Then, from Fig. 131,

$$\cos \alpha = \frac{b}{r}, \quad r = \sqrt{y^2 + b^2}, \quad dm = \delta \, dy.$$

$$\therefore F_x = \int_{-a}^{a} \frac{kb \, \delta \, dy}{(y^2 + b^2)^{\frac{3}{2}}} = \frac{2\,k\delta}{b} \frac{a}{\sqrt{a^2 + b^2}} = \frac{2\,k\delta}{b} \sin \omega.$$

The mass being homogeneous, the density is the mass per unit length; that is, $\delta = \frac{m}{2a}\cdot$ Hence we may write also

$$F_x = \frac{km}{b\sqrt{a^2+b^2}} = \frac{km}{ab}\sin\omega.$$

From symmetry it is obvious that $F_y = 0$. Hence F_x is the entire force of attraction upon the unit mass at O.

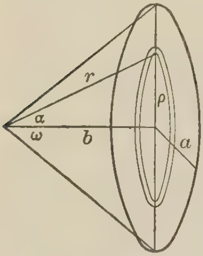

FIG. 132

Ex. 2. Find the attraction of a homogeneous flat disk of mass m and radius a upon a unit mass in the perpendicular through the center at a distance b from the latter.

Take for element of mass a ring of the disk of inner radius ρ and breadth $d\rho$. Then, if δ is the density of the disk and c its thickness (c is supposed to be so small that it is negligible in comparison with the other lengths involved; thus, in the expressions for r and $\cos\alpha$, c is taken equal to zero),

$$\cos\alpha = \frac{b}{r}, \quad r = \sqrt{\rho^2+b^2}, \quad dm = 2\pi\delta c\,\rho d\rho.$$

$$\therefore F_x = \int\frac{k\cos\alpha\,dm}{r^2} = 2\pi kc\,b\delta\int_0^a \frac{\rho d\rho}{(\rho^2+b^2)^{\frac{3}{2}}}$$

$$= 2\pi kc\delta\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right) = 2\pi kc\delta(1-\cos\omega).$$

Or, since $\delta = \frac{m}{\pi a^2 c}$, we have also

$$F_x = \frac{2km}{a^2}\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right) = \frac{2km}{a^2}(1-\cos\omega).$$

From symmetry

$$F_y = F_z = 0.$$

Ex. 3. Find the attraction of a homogeneous cone upon a unit mass at its vertex if its base radius is a and its altitude is h.

Consider the cone made up of slices, of thickness dx, made by planes parallel to the base. The attraction due to a slice at a distance x from the vertex is found from the first form of the attraction of a disk given in Ex. 2. In the present instance

$$c = dx, \quad b = x, \quad a = \frac{ax}{h}.$$

Hence for the entire cone

$$F_x = 2\pi k\delta \int^h \left(1 - \frac{h}{\sqrt{a^2 + h^2}}\right) dx = 2\pi k\delta h\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right)$$
$$= 2\pi k\delta h(1 - \cos\omega),$$

where ω is the angle which the generators of the cone make with the axis. Here $\delta = \dfrac{m}{\dfrac{\pi a^2 h}{3}}$. Hence we may write also

$$F_x = \frac{6\,km}{a^2}\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right) = \frac{6\,km}{a^2}(1 - \cos\omega).$$

In this case the attracted unit mass lies on the attracting cone. In formula (1), the integrand becomes infinite when $x = 0$, for then $r = 0$. We have here an illustration of a definite integral having a finite value even though the integrand becomes infinite at a point of the region over which the integral is taken.

EXERCISES

Find the attractions of the following bodies:

1. A homogeneous wire of mass m and length h upon a unit mass in the line of the wire at a distance b from one end.

2. A homogeneous wire of mass m and length h upon a unit mass not in the line of the wire, but at a distance b from the latter. Express the answer in terms of the angles which the lines joining the particle with the ends of the wire make with the perpendicular from the particle to the line of the wire.

3. A homogeneous wire of mass m bent in the form of an arc of a circle of radius a upon a unit mass at its center, (a) if the angle subtended by the arc is 2ω; (b) if the arc is a semicircle.

4. A homogeneous cylinder of mass m, base radius a, and altitude h, upon a unit mass, (a) on the axis of the cylinder at a distance b from one of its bases; (b) at the center of one of its bases.

CHAPTER XV

TAYLOR'S THEOREM AND APPLICATIONS

141. Rolle's Theorem. *If* $f(x)$ *and* $f'(x)$ *are continuous in the interval* $(a,\ b)$, *and if* $f(a)=f(b)=0$, *then* $f'(x)$ *vanishes for at least one value of* x *between* a *and* b.

In the special case when $f(x)$ has the constant value zero for all values of x in the interval $(a,\ b)$, $f'(x)=0$ for *all* values of x in this interval, and the theorem holds. Excluding this case, $f(x)$ will be positive or negative in some portion of the interval. If it is positive, it must have a maximum at some point in the interval since it is continuous. Similarly, if it is negative, it must have a minimum somewhere in the interval. In either case $f'(x)$ must vanish somewhere between a and b, since it is supposed to be continuous in the interval $(a,\ b)$. In Fig. 133, $f'(x)$ vanishes twice between $x=a$ and $x=b$.

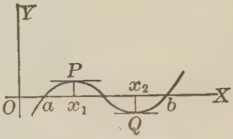

FIG. 133

The assumption that both $f(x)$ and $f'(x)$ are continuous in the interval $(a,\ b)$ is necessary, as Figs. 134 and 135 illustrate.

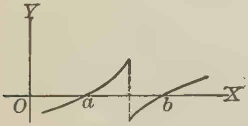

FIG. 134

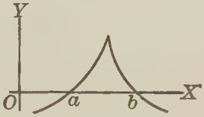

FIG. 135

The conclusion in Rolle's[1] theorem holds equally well whenever $f(a)=f(b)$, even when this common value is not zero. What we have established here may be expressed geometrically: *if* $f(x)$ *and* $f'(x)$ *are continuous throughout the interval* (a, b), *the curve* $y=f(x)$ *is such, that along any arc of it subtended by a horizontal chord (whose extremities have their abscissas in this interval) there is always a point at which the tangent to the curve is horizontal.*

---

[1] Michel Rolle (1652–1719) was a French mathematician.

142. Mean value theorem. *If* $f(x)$ *and* $f'(x)$ *are continuous, in the interval* (a, b), *there is at least one value* x_1 *of* x, *between* a *and* b, *such that*

$$\frac{f(b)-f(a)}{b-a}=f'(x). \tag{1}$$

The function

$$\phi(x)\equiv f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

and its derivative

$$\phi'(x)\equiv f'(x)-\frac{f(b)-f(a)}{b-a}$$

are also continuous from $x=a$ to $x=b$. Moreover

$$\phi(a)=\phi(b)=0.$$

Hence, by Rolle's theorem, $\phi'(x)$ vanishes for some value x_1 between a and b; that is,

$$f'(x_1)-\frac{f(b)-f(a)}{b-a}=0, \text{ where } a<x_1<b;$$

which proves the theorem.

The function $\phi(x)$ has a geometrical significance:

In Fig. 136 the curve AB represents $y=f(x)$, A being the point $(a, f(a))$ and B the point $(b, f(b))$. The equation of the chord AB, whose slope is $\frac{f(b)-f(a)}{b-a}$ is

$$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)$$

or

$$y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a).$$

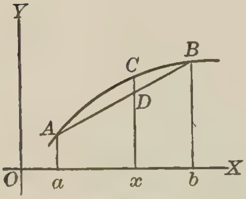

FIG. 136

Therefore CD, *the difference between the ordinates to the curve and to the chord corresponding to a general value of* x, *is*

$$f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a),$$

which is the function $\phi(x)$. It is zero obviously at A and at B.

The geometrical interpretation of the mean value theorem is that *if* $f(x)$ *and* $f'(x)$ *are continuous throughout the interval* (a, b) *the curve* $y=f(x)$ *is such that along any arc of it subtended by a chord (whose extremities have their abscissas in this interval) there is always a point at which the tangent to the curve is parallel to the chord.*

This is evidently a generalization of the geometrical interpretation of Rolle's theorem (§ 141).

Clearing of fractions, formula (1) may be written

$$f(b)-f(a)=(b-a)f'(x_1). \tag{2}$$

In this form the mean value theorem gives the change in value of the function corresponding to a finite change, $b-a$, in the variable. If $a+h$ is a value of x in the interval (a, b) we have

$$f(a+h)-f(a)=hf'(a+\theta h) \tag{3}$$

where $0<\theta<1$; for then $a<a+\theta h<a+h$.

In the form (3) or (2) our theorem is frequently referred to as the *theorem for finite increments.*

143. Extended mean value theorem. *If two functions* $f(x)$ *and* $\phi(x)$ *and their derivatives are continuous, and* $\phi'(x)\neq 0$ *throughout the interval* (a, b), *there is at least one value* x_1 *of* x *between* a *and* b *such that*

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(x)}{\phi'(x)}. \tag{1}$$

This extension of the mean value theorem is due to Augustine Louis Cauchy (1789–1857), one of France's most distinguished mathematicians, and is often referred to as Cauchy's formula. A simple proof for it, suggested by Professor A. A. Bennett in the *American Mathematical Monthly*, 1924, Vol. XXXI, p. 41, may be based upon the geometrical interpretation of the mean value theorem given in § 142.

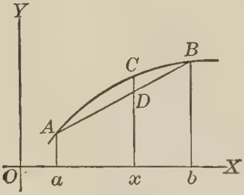

FIG. 136

If the equation of the curve AB in Fig. 136 is given parametrically,

$$x = \phi(t), \quad y = f(t),$$

where $\phi(\alpha) = a$ and $\phi(\beta) = b$,

the requirements of the mean value theorem are that $\phi(t)$, $f(t)$, $\phi'(t)$, and $f'(t)$ are continuous and $\phi'(t) \neq 0$ for $\alpha \leqq t \leqq \beta$. The geometrical interpretation of the mean value theorem, when expressed in terms of t, is then

$$\frac{f(\beta) - f(\alpha)}{\phi(\beta) - \phi(\alpha)} = \frac{f'(t_1)}{\phi'(t_1)}, \text{ where } \alpha < t_1 < \beta. \tag{2}$$

This is precisely the extended mean value theorem, in which the variable is t instead of x and the limits of the interval are α and β instead of a and b.

144. Taylor's theorem. The theorem for finite increments is a special case of a most important one known as *Taylor's theorem.*

If $f(x)$ and all its derivatives involved are continuous from $x = a$ to $x = a + h$, let us find an expression for the number P such that

$$f(a+h) - f(a) - hf'(a) - \frac{h^2}{2!}f''(a) - \cdots - \frac{h^n}{n!}f^{(n)}(a) = \frac{h^{n+1}}{(n+1)!}P, \tag{1}$$

where n is a positive integer.

Consider the auxiliary function

$$\phi(x) \equiv f(a+h) - f(x) - \frac{a+h-x}{1} f'(x) - \frac{(a+h-x)^2}{2\,!} f''(x)$$
$$- \cdots - \frac{(a+h-x)^n}{n\,!} f^{(n)}(x) - \frac{(a+h-x)^{n+1}}{(n+1)\,!} P,$$

whose derivative is

$$\phi'(x) \equiv - \frac{(a+h-x)^n}{n\,!} f^{(n+1)}(x) + \frac{(a+h-x)^n}{n\,!} P.$$

Both these functions are continuous from $x = a$ to $x = a + h$. Besides $\phi(a) = 0$ from (1) and $\phi(a+h) = 0$ by inspection. Hence $\phi'(a + \theta h) = 0$ by Rolle's theorem. That is,

$$P = f^{(n+1)}(a + \theta h). \tag{2}$$

Letting $R_n \equiv \frac{h^{n+1}}{(n+1)\,!} P = \frac{h^{n+1}}{(n+1)\,!} f^{(n+1)}(a + \theta h)$, we have established

Taylor's Theorem. *If $f(x)$ and all its derivatives involved are continuous from $x = a$ to $x = a + h$,*

$$f(a+h) = f(a) + \frac{h}{1\,!} f'(a) + \frac{h^2}{2\,!} f''(a) + \cdots + \frac{h^n}{n\,!} f^{(n)}(a) + R_n \tag{3}$$

where R_n is known as the remainder after the term in h^n. It is given by the formula

$$R_n = \frac{h^{n+1}}{(n+1)\,!} f^{(n+1)}(a + \theta h),\ 0 < \theta < 1. \tag{4}$$

This form of Taylor's theorem was first established by Joseph Louis Lagrange (1736–1813), another one of France's most distinguished mathematicians. The form (4) for R_n is known as *Lagrange's form of the remainder.*

To obtain a second form for R_n which is valuable, especially when Lagrange's form fails to give the desired information, we proceed as follows:

In (1) replace the right-hand member by hP. Employing the same reasoning as before, we find

$$R_n = \frac{h^{n+1}(1-\theta)^n}{n\,!} f^{(n+1)}(a + \theta h),\ 0 < \theta < 1, \tag{5}$$

which is known as *Cauchy's form of the remainder.*

Putting $h = x - a$, we have the useful form of Taylor's theorem,

$$f(x) = f(a) + \frac{f'(a)}{1\,!}(x-a) + \frac{f''(a)}{2\,!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n\,!}(x-a)^n + R_n, \tag{6}$$

where Lagrange's form of R_n is

$$\frac{(x-a)^{n+1}}{(n+1)\,!} f^{(n+1)}[a + \theta(x-a)],$$

and Cauchy's form of R_n is

$$\frac{(x-a)^{n+1}(1-\theta)^n}{n\,!} f^{(n+1)}[a + \theta(x-a)].$$

The right-hand member of (6) is often referred to as the *Taylor expansion* of $f(x)$ at $x = a$.

For the special value of $a = 0$ Taylor's theorem takes the form

$$f(x) = f(0) + \frac{f'(0)}{1\,!}x + \frac{f''(0)}{2\,!}x^2 + \cdots + \frac{f^{(n)}(0)}{n\,!}x + R_n, \tag{7}$$

where Lagrange's form of R_n is

$$\frac{x^{n+1}}{(n+1)\,!} f^{(n+1)}(\theta x),$$

and Cauchy's form of R_n is

$$\frac{x^{n+1}(1-\theta)^n}{n\,!} f^{(n+1)}(\theta x).$$

This special form of Taylor's theorem is often referred to as *Maclaurin's theorem.* But it should be borne in mind that the general term Taylor's theorem includes it.

Brook Taylor (1685–1731) was an Englishman. Colin Maclaurin (1698–1746) was a Scotchman.

145. Approximate value of a function. While the value of R_n in Taylor's theorem involves the quantity θ, whose actual value is usually not known, it is often practicable in the case of a given function to select a and n, so that, for all possible values of θ subject to the condition that $0 < \theta < 1$, the absolute value of R is less than an assigned positive value ϵ. In such a case, the error committed by replacing the Taylor expansion of the function by the polynomial arising when the remainder is omitted will be less than the assigned value ϵ. Evidently the smaller $x - a$ is, the smaller n may be. The choice of a is also largely determined by the fact that the values of the successive derivatives of $f(x)$ for $x = a$ appear in Taylor's formula. The following exercises will illustrate.

Ex. 1. Find the value of sin 31° correct to four decimal places. In this case the permissible error is less than 0.00005.

Since $D_x^n \sin x = \sin\left(\frac{n\pi}{2} + x\right)$, the Lagrange form of the remainder is

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!} \sin\left[\frac{(n+1)\pi}{2} + a + \theta(x-a)\right].$$

Using the notation $|A|$ for the absolute value of the quantity A

$$|R_n| = \frac{|x-a|^{n+1}}{(n+1)!} \left|\sin\left[\frac{(n+1)\pi}{2} + a + \theta(x-a)\right]\right|, \tag{1}$$

since the absolute value of the product of several factors is equal to the product of the absolute values of the factors. But for all values of x, $|\sin x| \overline{<} 1$. Hence from (1)

$$|R_n| \overline{<} \frac{|x-a|^{n+1}}{(n+1)!}. \tag{2}$$

In our exercise a good choice for a is 30°, or $\frac{\pi}{6}$ in radian measure. Then $x - a = \frac{\pi}{180} = 0.017453$, and

$$|R_2| \overline{<} \frac{(0.017)^3}{6} = 0.0000008.$$ Hence if we take

$$\sin a + \frac{\cos a}{1\,!}(x-a) - \frac{\sin a}{2\,!}(x-a)^2,$$

as an approximate value of $\sin x$, we have the value of $\sin 31°$, to the required degree of accuracy, given by

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{180} - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 = 0.5150.$$

Ex. 2. Find the value of $\sqrt{e} = e^{\frac{1}{2}}$, correct to five decimal places.

We shall take $a = 0$.

Since $D_x^n e^x = e^x$, the Lagrange form of the remainder is

$$R_n = \frac{x^{n+1}}{(n+1)\,!}e^{\theta x}, \text{ where } x = \tfrac{1}{2}.$$

It is known that $e < 4$. $\therefore e^{\theta x} < 2$ for all permissible values θ.

$$\therefore R_n < \frac{2(\frac{1}{2})^{n+1}}{(n+1)\,!} = \frac{1}{2^n(n+1)\,!}.$$

For $n = 6$, $R_n < 0.0000031$. Hence the required value of $\sqrt{e}$ is given by

$$1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \frac{x^6}{6\,!}$$

when $x = \frac{1}{2}$. Using it, we find 1.64872.

Ex. 3. Consider the behavior of the remainder for various allowable values of x in the Taylor expansion for $f(x) = \log(1+x)$.

Here

$$f'(x) = \frac{1}{1+x},\ f''(x) = \frac{-1}{(1+x)^2},\ f'''(x) = \frac{2\,!}{(1+x)^3},\ \cdots,$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)\,!}{(1+x)^n}.$$

It is easy to evaluate these for $x = 0$. Hence we shall take $a = 0$. Using Lagrange's form of the remainder, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{n+1}\frac{1}{(1+\theta x)^{n+1}}.$$

The absolute value of the remainder is

$$|R_n| = \frac{|x|^{n+1}}{n+1}\frac{1}{|1+\theta x|^{n+1}}. \tag{3}$$

If $|x| > 1$, the remainder is never negligible, for it grows in absolute value with n for sufficiently small values of θ.

If $0 < x \leq 1$, the numerator is never greater than 1 and the denominator grows with n. For a given positive value of x not greater than 1,

$$|R_n| < \frac{x^{n+1}}{n+1}, \tag{4}$$

since $(1+\theta x)^{n+1} > 1$.

But if $-1 < x < 0, 1 + \theta x < 1$, and since θ is not known exactly, one cannot tell from (3) how to calculate n so that $|R_n|$ is surely less than a given quantity for a given value of x.

In order to do this the remainder in the Cauchy form must be used. It is

$$R_n = \frac{(-1)^n x^{n+1}(1-\theta)^n}{(1+\theta x)^{n+1}} = (-1)^n\left(\frac{1-\theta}{1-\theta x}\right)^n \frac{x^{n+1}}{1+\theta x}.$$

This form of the remainder can be used for any value of x such that $-1 < x < 1$. Then

$$\frac{1-\theta}{1+\theta x} < 1 \quad \text{and} \quad 1 + \theta x \gtrless 1 - |x|.$$

$$\therefore |R_n| = \left|\frac{1-\theta}{1+\theta x}\right|^n \frac{|x|^{n+1}}{1+\theta x} < \frac{|x|^{n+1}}{1-|x|}. \tag{5}$$

For a given x, an n can be determined such that $|R_n|$ is less than a given number.

But this form fails to apply when $x = 1$, in which case the Lagrange form was applicable. Hence we see the need of having the two forms for the remainder.

Of course neither form is usable when $x = -1$, for $f(-1) = \log 0 = -\infty$.

EXERCISES

Find n so that the error committed by omitting R_n in the Taylor development of the following functions is less than 0.00005, and determine the corresponding approximate values:

1. $\cos 62°$. **2.** $\tan 44°$. **3.** $\tan^{-1}\frac{1}{10}$. **4.** e^2. **5.** $\log \frac{9}{10}$.

146. Taylor's series. The indicated sum of an indefinite number of terms, such as

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{1}$$

is known as an *infinite series.* The terms may be constants or functions of one or more variables. Thus

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \cdots + \frac{(-1)^{n+1}}{2} + \cdots, \tag{2}$$

$$a + ax + ax^2 + \cdots + ax^{n-1} + \cdots, \tag{3}$$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \tag{4}$$

are examples of infinite series. It is obviously impossible to write all the terms of an infinite series. All that is presupposed is that the law of the terms of the series is known, so that it is possible to write the nth term for any given n, no matter how large. This is obviously true of the series (2), (3), (4).

Let s_n be the sum of the first n terms of an infinite series. If as n increases indefinitely s_n approaches a definite finite limit S, that is

$$\underset{n \to \infty}{L}\, s_n = S,$$

this limit is defined as the *sum* of the infinite series and the series is said to be *convergent.* The difference $S - s_n \equiv r_n$ is said to be the *remainder* of the series after the nth term. It also is an infinite series, formed of the terms of (1) following the nth term u_n. *The remainder, r_n, is an infinite series, which, in case of a convergent series, approaches 0 as n grows indefinitely;* for $r_n = S - s_n$, and $\underset{n \to \infty}{L} r_n = \underset{n \to \infty}{L} (S - s_n)$. But S is constant, and $\underset{n \to \infty}{L} s_n = S$. $\therefore \underset{n \to \infty}{L} (S - s_n) = 0$.

Moreover the condition $\underset{n \to \infty}{L} r_n = 0$ is sufficient for the convergence of the series.

Consider series (3). It is known as a *geometric progression,* characterized by the fact that the ratio of each term to the preceding one is the same for all values of n. In this case the common ratio is x. To sum this series we note that

$$s_n = a + ax + ax^2 + \cdots + ax^{n-1}. \tag{5}$$

Multiplying both members of (5) by x, we have

$$xs_n = ax + ax^2 + \cdots + ax^{n-1} + ax^n. \tag{6}$$

Subtracting (6) from (5),

$$(1 - x)s_n = a - ax^n. \quad \therefore s_n = \frac{a(1 - x^n)}{1 - x}.$$

If $|x| < 1$, $|x|^n$ diminishes as n increases, and $\underset{n \to \infty}{L} x^n = 0$, Hence, in this case, the series (3) is convergent and its sum is

$$S = \underset{n \to \infty}{L} s_n = \frac{a}{1 - x}. \tag{7}$$

Here
$$r_n = S - s_n = \frac{ax^n}{1 - x}.$$

If $|x| > 1$, $|s_n|$ grows indefinitely with n, since $|x|^n$ does. For such values of x the series is *divergent.*

If $x = 1$, $s_n = na$, which also grows indefinitely with n, and the series is divergent.

If $x = -1$, $s_{2n} = 0$, $s_{2n-1} = a$. Hence $\underset{n \to \infty}{L} s_n$ does not exist, and the series is said to be divergent in this case also.

Hence only when the common ratio is less than 1 in absolute value is a geometric progression convergent. And its value is given by (7).

Series (2) also is a geometric progression, in which the common ratio is $-\frac{1}{2}$. Hence it is convergent and has the sum

$$S = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}.$$

Series (4) is not a geometric progression.[1] Nor is it easy to find an expression for s_n, the sum of the first n terms. But if we expand e^x by Maclaurin's theorem we find

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_{n-1}$$

where $R_{n-1} = \frac{x^n}{n!} e^{\theta x}$, $0 < \theta < 1$, using the Lagrange form.

For a given value of x, e^x has a given value, say M. While M may be a large number when x is large, it is still a definite finite number and $e^{\theta x} < M$.

Moreover for a given value of x, $\frac{x^n}{n!}$ diminishes towards zero as n grows.[2] Hence R_n approaches zero as n increases; and the series (4) is convergent. Of course, for large values of x, n must be taken a large number before R_n becomes less than an assigned small number. We say that for large values of x the series *converges slowly*. But it converges for any given value of x.

---

[1] Rules for the convergence and divergence of many infinite series which are not geometric progressions are known and may be found in books dealing with College Algebra. They will not be taken up here.

[2] It is true that, for x a given large number, $\frac{x^n}{n!}$ starts out by growing with n. But after $n > x$, it diminishes more and more rapidly as n increases.

Whenever, in the Taylor expansion of a function $f(x)$, it is possible to name a positive number m so that, corresponding to a given value of x, R_n is less than any assigned positive number, no matter how small, for $n \overline{>} m$, we shall say that the value of $f(x)$ is given by the infinite series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots, \qquad (8)$$

which is known as a *Taylor's series.* In particular if $a = 0$ we have the *Maclaurin's series*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \qquad (9)$$

In employing a Taylor's or Maclaurin's series to represent a function, care must be exercised to restrict the variable to such values for which the series converges. No general rule can be given here. The allowable values of the variable for which the series, given in this section, may be used will be indicated, excepting when there is no restriction, in which case the variable may take any real value.

Ex. 1. Find the Maclaurin's series for $f(x) = \sqrt{1-x}$.

Writing $f(x) = (1-x)^{\frac{1}{2}}$, we have $f(0) = 1$, and on differentiating

$$f'(x) = -\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \qquad \therefore f'(0) = -\tfrac{1}{2},$$

$$f''(x) = -\frac{1}{2^2}(1-x)^{-\frac{3}{2}} \qquad \therefore f''(0) = -\frac{1}{2^2},$$

$$f'''(x) = -\frac{1\cdot 3}{2^3}(1-x)^{-\frac{5}{2}} \qquad \therefore f'''(0) = -\frac{1\cdot 3}{2^3},$$

. . . .

$$f^{(n)}(x) = -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}(1-x)^{-\frac{2n-1}{2}}.$$

$$\therefore f^{(n)}(0) = -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}.$$

Substituting these in Maclaurin's series (9) we have

$$\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{2^2 2!}-\frac{1\cdot 3}{2^3 3!}x^3 - \cdots - \frac{1\cdot 3\cdot 5\cdots 2n-3}{2^n n!}x^n - \cdots.$$

This series fails to represent the function when $|x|>1$. It can be shown that the series converges for $x=\pm 1$ even though $f'(1)=\infty$.

The student will recognize this as the same expansion as is given by the binomial theorem.

Ex. 2. Find the Taylor's series for $\cos x$ when $a=\frac{\pi}{2}\cdot$

Writing $f(x)=\cos x$, we have $f\left(\frac{\pi}{2}\right)=0$, and, on differentiating,

$$f'(x)=-\sin x \qquad \therefore f'\left(\frac{\pi}{2}\right)=-1,$$

$$f''(x)=-\cos x \qquad \therefore f''\left(\frac{\pi}{2}\right)=0,$$

$$f'''(x)=\sin x \qquad \therefore f'''\left(\frac{\pi}{2}\right)=1,$$

. . . .

$$f^{(n)}(x)=\cos\left(\frac{n\pi}{2}+x\right). \qquad \therefore f^{(n)}\left(\frac{\pi}{2}\right)=\cos\,(n+1)\frac{\pi}{2}\cdot$$

Substituting these in Taylor's series (8) we have

$$\cos x=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!} + \cdots + \frac{\cos\,(n+1)\frac{\pi}{2}}{n!}\left(x-\frac{\pi}{2}\right)^n+\cdots.$$

EXERCISES

Verify the following expansions and memorize the first five.

1. $e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots + \frac{x^n}{n\,!} + \cdots.$

2. $\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \cdots.$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1}x^{2n}}{(2n)!} + \cdots.$

4. $\log(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots$, if $-1 < x \leq 1$.

5. $(1+x)^r = 1 + rx + \frac{r(r-1)}{2\,!}x^2 + \cdots + \frac{r(r-1)(r-2)\cdots(r-n+1)}{n\,!}x^n + \cdots,$
if $|x| < 1$.

6. $\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, if $|x| \leq 1$.

7. $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2\,x^6}{45} - \frac{x^8}{315} + \cdots + \frac{(-1)^{n-1}(2\,x)^{2n}}{2(2\,n)\,!} + \cdots.$

HINT. $\sin^2 x = \frac{1 - \cos 2\,x}{2}.$

8. $e^x \sin x = x + \frac{2\,x^2}{2\,!} + \frac{2\,x^3}{3\,!} - \frac{4\,x^5}{5\,!} - \frac{8\,x^6}{6\,!} - \frac{8\,x^7}{7\,!} + \cdots.$

9. $e^x \cos x = 1 + x - \frac{2\,x^3}{3\,!} - \frac{4\,x^4}{4\,!} - \frac{4\,x^5}{5\,!} + \frac{8\,x^7}{7\,!} + \cdots.$

10. $\log x = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$
$+ \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots,$
if $-1 < (x - 1) \leq 1$ or $0 < x \leq 2$.

11. $\log x = \log a + \frac{x-a}{a} - \frac{(x-a)^2}{2\,a^2} + \cdots + (-1)^{n-1}\frac{(x-a)^n}{na^n} + \cdots,$
if $-1 < (x - a) \leq 1$ or $a - 1 < x \leq a + 1$.

12. $$\sqrt{x} = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{2^2\, 2\,!} + \frac{1 \cdot 3}{2^3\, 3\,!}(x-1)^3 - \frac{1 \cdot 3 \cdot 5}{2^4\, 4\,!}(x-1)^4 + \cdots, \text{ if } |x-1| \leq 1.$$

13. $$\sqrt{x} = a^{\frac{1}{2}} + \frac{x-a}{2\,a^{\frac{1}{2}}} - \frac{(x-a)^2}{2^2\, 2\,!\, a^{\frac{3}{2}}} + \frac{1 \cdot 3(x-a)^3}{2^3\, 3\,!\, a^{\frac{5}{2}}} - \cdots, \quad \text{if } |x-a| < |a|.$$

14. $$2^x = 1 + \frac{x \log 2}{1} + \frac{(x \log 2)^2}{2\,!} + \cdots + \frac{(x \log 2)^n}{n\,!} + \cdots.$$

147. Euler's formula. When the Taylor's series representing a function is convergent for a set of imaginary values of the variable, the series may be taken as the definition of the function for these values. Frequently the original definition of a function fails when the variable is not real. Thus "*e* raised to an imaginary power" has no meaning. If in the Taylor (or Maclaurin) development

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n\,!} + \cdots, \tag{1}$$

we let $x = iu$, where $i = \sqrt{-1}$, we have

$$e^{iu} = 1 + \frac{iu}{1!} + \frac{(iu)^2}{2!} + \frac{(iu)^3}{3\,!} + \cdots + \frac{(iu)^n}{n\,!} + \cdots. \tag{2}$$

This series can be shown to be convergent for all values of u. Hence it may be taken as the definition of e^{iu}. Noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on, we may rewrite (2), separating the real and imaginary terms,

$$e^{iu} = 1 - \frac{u^2}{2\,!} + \frac{u^4}{4\,!} - \frac{u^6}{6\,!} + \cdots + i\left(\frac{u}{1} - \frac{u^3}{3\,!} + \frac{u^5}{5\,!} - \frac{u^7}{7\,!} + \cdots\right). \tag{3}$$

Recalling the Taylor (or Maclaurin) expansion for $\cos u$ and $\sin u$, we have then

$$\boldsymbol{e^{iu} = \cos u + i \sin u.} \tag{4}$$

This formula is due to Leonard Euler (1707–1783), one of the world's greatest mathematicians, born in Basle, Switzerland.

If u is replaced by $-u$, (4) leads to

$$e^{-iu} = \cos u - i \sin u, \tag{5}$$

since $\cos(-u) = \cos u$ and $\sin(-u) = -\sin u$.

Solving (4) and (5) for $\sin u$ and $\cos u$, we have

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i}, \quad \cos u = \frac{e^{iu} + e^{-iu}}{2}. \tag{6}$$

148. Indeterminate forms. We shall consider in this section the limits towards which certain types of functions approach as the variable approaches certain critical values at which the functions are absolutely indeterminate. Thus we have already seen that $\frac{\sin\theta}{\theta}$, which is $\frac{0}{0}$ when $\theta = 0$, approaches 1 as θ approaches 0; while $\frac{(1-\cos\theta)}{\theta}$, which also has the indeterminate form $\frac{0}{0}$ when $\theta = 0$, approaches 0 as θ approaches 0; and we also noted that $\left(1 + \frac{1}{x}\right)^x$, which has the indeterminate form 1^∞ when $x = \infty$, approaches e as x approaches ∞.

Case 1: $\mathbf{\frac{0}{0}}$. If $f(x)$ and $\phi(x)$ are continuous from $x = a$ to $x = a + h$, and have continuous derivatives at each point of the interval, then using Taylor's series

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a) + h^2\frac{f''(a)}{2!} + \cdots}{\phi(a) + h\phi'(a) + h^2\frac{\phi''(a)}{2!} + \cdots}.$$

If $f(a) = 0$ and $\phi(a) = 0$, we have, after dividing by h,

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f'(a) + h\frac{f''(a)}{2!} + \cdots}{\phi'(a) + h\frac{\phi''(a)}{2!} + \cdots}. \tag{1}$$

And as h approaches 0

$$\underset{h \to 0}{L} \frac{f(a+h)}{\phi(a+h)} = \frac{f'(a)}{\phi'(a)}. \qquad (2)$$

If, however, $f'(a)$ and $\phi'(a)$ are also equal to 0, we have from (1), after striking out the common factor $\frac{h}{2!}$

$$\underset{h \to 0}{L} \frac{f(a+h)}{\phi(a+h)} = \frac{f''(a)}{\phi''(a)}. \qquad (3)$$

And so on.

Ex. 1. Find $\underset{x \to 0}{L} \frac{\sin x}{x}$.

Formula (2) applies.

$$\therefore \underset{x \to 0}{L} \frac{\sin x}{x} = \left.\frac{\cos x}{1}\right]_0 = \frac{1}{1} = 1.$$

Ex. 2. Find $\underset{x \to 0}{L} \frac{x - \sin x}{x^3}$.

$\underset{x \to 0}{L} \frac{x - \sin x}{x^3} = \left.\frac{1 - \cos x}{3x^2}\right]_0 = \frac{0}{0}$. Hence we differentiate numerator and denominator again, obtaining

$\underset{x \to 0}{L} \frac{x - \sin x}{x^3} = \left.\frac{\sin x}{6x}\right]_0 = \frac{0}{0}$. Differentiating again

$$\underset{x \to 0}{L} \frac{x - \sin x}{x^3} = \left.\frac{\cos x}{6}\right]_0 = \frac{1}{6}.$$

Ex. 3. Find $\underset{x \to \frac{\pi}{4}}{L} \frac{1 - \tan x}{\cos^2 2x}$.

$$\underset{x \to \frac{\pi}{4}}{L} \frac{1 - \tan x}{\cos^2 2x} = \left.\frac{-\sec^2 x}{-2\cos 2x \sin 2x}\right]_{\frac{\pi}{4}} = \frac{2}{0} = \infty.$$

EXERCISES

Find the following limits:

1. $\underset{x \to 0}{L} \dfrac{1 - \cos x}{\sin x}.$

2. $\underset{x \to 1}{L} \dfrac{x^5 - 1}{x^2 - 1}.$

3. $\underset{x \to 0}{L} \dfrac{\tan x - x}{x - \sin x}.$

4. $\underset{x \to 0}{L} \dfrac{\log \cos x}{x}.$

5. $\underset{x \to \frac{\pi}{2}}{L} \dfrac{1 + \cos x - \sin x}{2 \sin x \cos x - \cos x}.$

6. $\underset{x \to a}{L} \dfrac{x^3 - a^3}{\sin (x - a)}.$

7. $\underset{x \to 0}{L} \dfrac{\tan x - \sin x}{x^2 \log (1 + x)}.$

8. $\underset{x \to 0}{L} \dfrac{a^x - b^x}{x}.$

9. $\underset{x \to 0}{L} \dfrac{e^{\tan x} - e^x}{\tan x - x}.$

10. $\underset{x \to 1}{L} \dfrac{\log x}{1 - x}.$

Case 2: $\frac{\infty}{\infty}$. If $f(x)$ and $\phi(x)$ approach ∞ as x approaches a in such a way that $\dfrac{1}{f(x)}$ and $\dfrac{1}{\phi(x)}$, as well as their derivatives, are continuous for $a < x \leq a + h$, and approach 0 as x approaches a, $\dfrac{f(x)}{\phi(x)}$ may be replaced by $\dfrac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}}$, which comes under case 1.

When this method is not practicable, use may be made of the fact that in this case also

$$\underset{x \to a}{L} \frac{f(x)}{\phi(x)} = \underset{x \to a}{L} \frac{f'(x)}{\phi'(x)}, \tag{4}$$

whenever $\underset{x \to a}{L} \dfrac{f'(x)}{\phi'(x)}$ exists, it being understood that, for a value of h that can be named, $f(x)$ and $\phi(x)$ are continuous and have continuous derivatives in the interval $a < x \leq a + h$, and that $f(x)$ and $\phi(x)$ do not vanish in this interval.

To prove this we make use of the extended mean value theorem (§ 143). Applying this we have

$$\frac{f(x) - f(a + h)}{\phi(x) - \phi(a + h)} = \frac{f'(x_1)}{\phi'(x_1)}, \text{ where } a < x < x_1 < a + h, \tag{5}$$

or what is the same thing

$$\frac{f(x)}{\phi(x)}\frac{\dfrac{1-f(a+h)}{f(x)}}{\dfrac{1-\phi(a+h)}{\phi(x)}}=\frac{f'(x_1)}{\phi'(x_1)},\quad \text{or}\quad \frac{f(x)}{\phi(x)}=\frac{\dfrac{1-\phi(a+h)}{\phi(x)}}{\dfrac{1-f(a+h)}{f(x)}}\frac{f'(x_1)}{\phi'(x_1)}. \qquad (7)$$

If $\underset{x\to a}{L}\dfrac{f'(x)}{\phi'(x)}$ exists, call it L. Then we may write $\dfrac{f'(x_1)}{\phi'(x_1)}=L+\alpha(x_1)$, where $\alpha(x_1)$ approaches 0 as x_1 approaches a. Moreover, since $f(x)$ and $\phi(x)$ approach infinity as x approaches a and are not zero in the interval under consideration, $\dfrac{\dfrac{1-\phi(a+h)}{\phi(x)}}{\dfrac{1-f(a+h)}{f(x)}}$ approaches 1 as x approaches a; that is, $\dfrac{\dfrac{1-\phi(a+h)}{\phi(x)}}{\dfrac{1-f(a+h)}{f(x)}}=1+\beta(x)$, where $\beta(x)$ approaches 0 as x approaches a.

Hence from (6),

$$\frac{f(x)}{\phi(x)}=(1+\beta)(L+\alpha)=L+\beta L+\alpha+\alpha\beta.$$

Passing to the limit,

$$\underset{x\to a}{L}\frac{f(x)}{\phi(x)}=L=\underset{x\to a}{L}\frac{f'(x)}{\phi'(x)}.$$

Here L was treated as a finite number. The argument can easily be modified for the case when $L=\infty$.

In the application of (4), it should be noted that usually $\dfrac{f'(x)}{\phi'(x)}$ also assumes the form $\frac{\infty}{\infty}$ for the critical value of x. We proceed with the finding of the limit of this as we would with the original function; that is, we examine $\dfrac{\dfrac{1}{\phi'(x)}}{\dfrac{1}{f'(x)}}$ by the method of case 1. If this does not work, use (4) a second time and proceed to find the limit of $\dfrac{f''(x)}{\phi''(x)}$. And so on.

Formula (4) still holds when $a=\infty$. For introducing the new variable $t = \frac{1}{x}$ which approaches 0 as x approaches a, we have

$$\underset{x \to \infty}{L} \frac{f(x)}{\phi(x)} = \underset{t \to 0}{L} \frac{f\left(\frac{1}{t}\right)}{\phi\left(\frac{1}{t}\right)} = \underset{t \to 0}{L} \frac{-\frac{1}{t^2} f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2} \phi'\left(\frac{1}{t}\right)} = \underset{t \to 0}{L} \frac{f'\left(\frac{1}{t}\right)}{\phi'\left(\frac{1}{t}\right)}$$

$$= \underset{x \to \infty}{L} \frac{f'(x)}{\phi'(x)}.$$

Ex. 4. Find $\underset{x \to 0}{L} \frac{\cot x}{\log x}$.

$$\underset{x \to 0}{L} \frac{\cot x}{\log x} = \underset{x \to 0}{L} \frac{-\csc^2 x}{\frac{1}{x}} = \underset{x \to 0}{L} \frac{x}{-\sin^2 x} = \left.\frac{1}{-2 \sin x \cos x}\right]_0$$

$$= -\infty.$$

Ex. 5. Find $\underset{x \to \frac{\pi}{2}}{L} \frac{\tan x}{\tan 3x}$.

$$\underset{x \to \frac{\pi}{2}}{L} \frac{\tan x}{\tan 3x} = \underset{x \to \frac{\pi}{2}}{L} \frac{\sec^2 x}{3 \sec^2 3x} = \underset{x \to \frac{\pi}{2}}{L} \frac{\cos^2 3x}{3 \cos^2 x} = \underset{x \to \frac{\pi}{2}}{L} \frac{1 + \cos 6x}{3(1 + \cos 2x)}$$

$$= \underset{x \to \frac{\pi}{2}}{L} \frac{-6 \sin 6x}{-6 \sin 2x} = \underset{x \to \frac{\pi}{2}}{L} \frac{6 \cos 6x}{2 \cos 2x} = 3.$$

Ex. 6. Find $\underset{x \to \infty}{L} \frac{e^{x^2}}{x^3}$.

$$\underset{x \to \infty}{L} \frac{e^{x^2}}{x^3} = \underset{x \to \infty}{L} \frac{2xe^{x^2}}{3x^2} = \underset{x \to \infty}{L} \frac{2e^{x^2}}{3x} = \underset{x \to \infty}{L} \frac{4xe^{x^2}}{3} = \infty.$$

EXERCISES

Find the following limits:

11. $\underset{x \to 0}{L} \frac{\log \sin 2x}{\log \sin x}$.

12. $\underset{x \to \infty}{L} \frac{x^2}{\log x}$.

13. $\underset{x \to \infty}{L} \frac{x^2}{e^x}$.

14. $\underset{x \to 1}{L} \frac{\sec \frac{\pi x}{2}}{\log (1 - x)}$.

15. $\underset{x \to \infty}{L} \dfrac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}$,

(a) if $n > m$; (b) if $n = m$; (c) if $n < m$.

Case 3: $0 \cdot \infty$. If one of the functions $f(x)$ and $\phi(x)$ approaches 0 and the other approaches ∞ as x approaches a, the limit of the product $f(x)\,\phi(x)$ may be determined by replacing the product by

$$\frac{f(x)}{\dfrac{1}{\phi(x)}} \quad \text{or} \quad \frac{\phi(x)}{\dfrac{1}{f(x)}},$$

which come under case 1 or case 2.

Ex. 7. Find $\underset{x \to 0}{L}\, x \log x$.

Writing $x \log x = \dfrac{\log x}{\dfrac{1}{x}}$, we have

$$\underset{x \to 0}{L}\, x \log x = \underset{x \to 0}{L} \frac{\log x}{\dfrac{1}{x}} = \underset{x \to 0}{L} \frac{\dfrac{1}{x}}{-\dfrac{1}{x_2}} = \underset{x \to 0}{L} (-x) = 0.$$

EXERCISES

Find the following limits:

16. $\underset{x \to 0}{L}\, x \cot \pi x$.

17. $\underset{x \to \frac{\pi}{2}}{L} \left(x - \dfrac{\pi}{2}\right) \tan x$.

18. $\underset{x \to 0}{L}\, x^2 e^{\frac{1}{x^2}}$, also $\underset{x \to \infty}{L}\, x^2 e^{-\frac{1}{x^2}}$.

19. $\underset{x \to 1}{L} \sec \dfrac{\pi x}{2} \log x$.

20. $\underset{x \to \infty}{L} \tan \dfrac{1}{x} \log x$.

Case 4: 1^∞, 0^0, ∞^0. The function $\phi(x)^{f(x)}$ has no meaning for $x = a$, for certain forms of $f(x)$ and $\phi(x)$. Writing

$$\phi(x) = e^{\log \phi(x)},$$

which follows from the definition of $\log \phi(x)$, we have

$$\phi(x)^{f(x)} = [e^{\log \phi(x)}]^{f(x)} = e^{f(x) \log \phi(x)}.$$

Now $f(x) \log \phi(x)$ assumes the indeterminate form $0 \cdot \infty$

(*a*) if $f(a) = \infty$ and $\phi(a) = 1$; *i.e.* if $\phi(a)^{f(a)}$ has the form 1^∞;

(*b*) if $f(a) = 0$ and $\phi(a) = 0$; *i.e.* if $(\phi)^{f(a)}$ has the form 0^0;

(*c*) if $f(a) = 0$ and $\phi(a) = \infty$; *i.e.* if $\phi(a)^{f(a)}$ has the form ∞^0.

To determine the limit of $\phi(x)^{f(x)}$, in any of these cases, as x approaches a, we let $u = \phi(x)^{f(x)}$, whence $\log u = f(x) \log \phi(x)$, and find $\underset{x \to a}{L} \log u$, which comes under case 3. Finally the limit of u is equal to $e^{\underset{x \to a}{L} \log u}$.

Ex. 8. Find $\underset{x \to 0}{L} (\cos x)^{\frac{1}{x^2}}$.

Let $u = (\cos x)^{\frac{1}{x^2}}$. Then $\log u = \dfrac{\log \cos x}{x^2}$.

$$\underset{x \to 0}{L} \frac{\log \cos x}{x^2} = \underset{x \to 0}{L} \frac{-\tan x}{2x} = \underset{x \to 0}{L} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

$$\therefore \underset{x \to 0}{L} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$$

Ex. 9. Find $\underset{x \to 0}{L} (\cos x)^{\frac{1}{x^3}}$.

Let $u = (\cos x)^{\frac{1}{x^3}}$. Then $\log u = \dfrac{\log \cos x}{x^3}$.

$$\underset{x \to 0}{L} \frac{\log \cos x}{x^3} = \underset{x \to 0}{L} \frac{-\tan x}{3x^2} = \underset{x \to 0}{L} \frac{-\sec^2 x}{6x} = -\infty \quad \text{or} \quad +\infty$$

depending upon whether x approaches 0 from the positive or negative side. Hence

$\underset{x \to 0}{L} (\cos x)^{\frac{1}{x^3}} = e^{-\infty} = 0$ when x approaches 0 from the positive side.

$\underset{x \to 0}{L} (\cos x)^{\frac{1}{x^3}} = e^{\infty} = \infty$ where x approaches 0 from the negative side.

We have here an example of a function which approaches different values as the variable approaches a critical value from different sides, and has no value when the variable assumes the critical value.

EXERCISES

Find the following limits:

21. $\underset{x \to 1}{L}\, x^{\frac{1}{1-x}}$.

22. $\underset{x \to 0}{L}\, (\cot x)^{\sin x}$.

23. $\underset{x \to 0}{L}\, (e^x + x)^{\frac{1}{x}}$.

24. $\underset{x \to 0}{L}\, (x + \cos x)^{\frac{1}{x}}$.

25. $\underset{x \to \infty}{L}\, x^{\frac{1}{x}}$.

26. $\underset{x \to \frac{\pi}{2}}{L}\, (\sin x)^{\tan x}$.

Case 5: $\infty - \infty$. If $f(x)$ and $\phi(x)$ both become ∞ when $x = a$, their difference $f(x) - \phi(x)$ has no meaning when $x = a$. Frequently this can be put into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and can then be handled by the appropriate method. The following exercises will illustrate.

Ex. 10. Find $\underset{x \to 0}{L}\, (\csc x - \cot x)$.

Writing $\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x}$, we have

$$\underset{x \to 0}{L}\, (\csc x - \cot x) = \underset{x \to 0}{L}\, \frac{1 - \cos x}{\sin x} = \underset{x \to 0}{L}\, \frac{\sin x}{\cos x} = 0.$$

Ex. 11. Find $\underset{x \to \infty}{L}\, (\sqrt{1 + x^2} - x)$.

Introduce the new variable $t = \frac{1}{x}$.

$$\sqrt{1 + x^2} - x = \frac{\sqrt{1 + t^2}}{t} - \frac{1}{t} = \frac{\sqrt{1 + t^2} - 1}{t}.$$

$$\underset{x \to \infty}{L}\, (\sqrt{1 + x^2} - x) = \underset{t \to 0}{L}\, \frac{\sqrt{1 + t^2} - 1}{t} = \underset{t \to 0}{L}\, \frac{\frac{t}{\sqrt{1 + t^2}}}{1} = 0.$$

Ex. 12. Find $\underset{x\longrightarrow 1}{L}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$.

$$\underset{x\longrightarrow 1}{L}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)=\underset{x\longrightarrow 1}{L}\frac{x\log x-x+1}{(x-1)\log x}=\underset{x\longrightarrow 1}{L}\frac{\log x+1}{\log x+1-\frac{1}{x}}$$

$$=\underset{x\longrightarrow 1}{L}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}.$$

EXERCISES

Find the following limits:

27. $\underset{x\longrightarrow\infty}{L}(\log(x+1)-\log x)$.

28. $\underset{x\longrightarrow 0}{L}\left(\csc^2 x-\frac{1}{x^2}\right)$.

29. $\underset{x\longrightarrow\frac{\pi}{2}}{L}(\sec x-\tan x)$.

30. $\underset{x\longrightarrow 0}{L}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

. . . .

31. $\underset{x\longrightarrow a}{L}\frac{e^x-e^a}{x-a}$.

32. $\underset{x\longrightarrow\frac{\pi}{4}}{L}\frac{\tan x-1}{x-\frac{\pi}{4}}$.

33. $\underset{x\longrightarrow\infty}{L}\frac{e^x}{x^n}$, $n>0$.

34. $\underset{x\longrightarrow\infty}{L}\frac{\log x}{x^n}$, $n>0$.

35. $\underset{x\longrightarrow 0}{L}\left(\cot^2 x-\frac{1}{x^2}\right)$.

36. $\underset{x\longrightarrow\infty}{L}xe^{-x}$.

37. $\underset{x\longrightarrow 0}{L}(1+\sin x)^{\cot x}$.

38. $\underset{x\longrightarrow 0}{L}\sin x\log\cot x$.

39. $\underset{x\longrightarrow 0}{L}(1+x^2)^{\frac{1}{x}}$.

40. $\underset{x\longrightarrow\infty}{L}(1+x^2)^{\frac{1}{x}}$.

41. $\underset{x\longrightarrow -1}{L}\frac{a^{x+1}-1}{x+1}$.

42. $\underset{x\longrightarrow 0}{L}\frac{\tan x-\sin x}{x^3}$.

43. $\underset{x\longrightarrow 0}{L}\left(\frac{2}{x}-\cot\frac{x}{2}\right)$.

44. $\underset{x\longrightarrow 0}{L}\left(\csc x-\frac{1}{x}\right)$.

45. $\underset{x\longrightarrow 0}{L}\frac{e^x-e^{-x}-2x}{x-\sin x}$.

46. $\underset{x\longrightarrow 0}{L}(\cos x)^{\frac{1}{x^4}}$.

47. $\underset{x\longrightarrow 1}{L}\tan\frac{\pi}{2}x\log(2-x)$.

48. $\underset{x\longrightarrow\infty}{L}x\sin\frac{a}{x}$.

49. $\underset{x\to 0}{L}\dfrac{\sin x - x\cos x}{x - \sin x}$.

50. $\underset{x\to 0}{L}(\cot x)^{\sin x}$.

51. $\underset{x\to 0}{L}\, x\log\sin x$.

52. $\underset{x\to 1}{L}(1-x)\sec\dfrac{\pi}{2}x$.

53. $\underset{x\to\infty}{L}\dfrac{\log x}{\log(\log x)}$.

54. $\underset{x\to 0}{L}\dfrac{\log x}{\log\sin x}$.

55. $\underset{x\to 1}{L}\dfrac{\log x}{1-x}$.

56. $\underset{x\to 0}{L}\, x^x$.

57. $\underset{x\to 2}{L}(x-2)\cot\pi x$.

58. $\underset{x\to 0}{L}(1+ax)^{\frac{1}{x}}$.

59. $\underset{x\to 0}{L}(\tan x)^{\sin x}$.

60. $\underset{x\to 0}{L}\left(\dfrac{1}{x}\right)^{\tan x}$.

149. Operations with Taylor series. It can be proved that if the Taylor series representing two functions

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \tag{1}$$

$$\phi(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots \tag{2}$$

are convergent in the same interval $(a-k,\ a+k)$, then (a) the sums of the corresponding terms of the two series form a new series which represents the sum of $f(x)$ and $\phi(x)$ in the interval $(a-k,\ a+k)$; (b) the differences of the corresponding terms of the two series form a series which represents the difference of the functions in the interval $(a-k,\ a+k)$; (c) the series, whose terms are obtained by taking the product of each term of (1) by each term of (2), and then arranged in powers of x, represents the product of the functions in the interval $(a-k,\ a+k)$; (d) the series obtained by dividing (1) by (2) represents the quotient $\dfrac{f(x)}{\phi(x)}$ and holds in the interval $(a-h,\ a+h)$, where h is either k or the distance of the nearest root of $\phi(x)=0$ from a, if this is less than k; (e) if the terms of the Taylor series are differentiated, the resulting series represents the derivative of the function represented by the original series, both series having the same interval of convergence; (*f*) if the terms of the Taylor series are integrated, the resulting series represents the integral of the function represented by the original series, both series having the same interval of convergence.

In other words, the Taylor series representing functions may be treated like polynomials, with the caution as to the interval over which the result holds in the case of division.

Ex. 1. Show that

$$\log\frac{1+x}{1-x}=2\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right),\ \text{for } -1<x<1.$$

We saw that

$$\log(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots,\ \text{for } -1<x\leq 1. \qquad (3)$$

If we put $x=-y$, this leads to

$$\log(1-y)=-\frac{y}{1}-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\cdots,$$

$$\text{for } -1<-y\overline{<}1 \text{ or } -1\leq y<1.$$

$$\therefore \log(1-x)=-\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots,\ \text{for } -1\leq x<1. \qquad (4)$$

Series (3) and (4) hold simultaneously in the interval $-1<x<1$. Hence we may take their difference, term for term. Noting that $\log(1+x)-\log(1-x)=\log\dfrac{1+x}{1-x}$, the desired result follows.

Ex. 2. Show that

$$\sec x=1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+\cdots,\ \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

We saw that

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots,\ \text{for all finite values of } x.$$

Hence to find the Maclaurin development for $\sec x$ we use (9) of § 146 or (*d*) above, in the case of $\dfrac{1}{\cos x}$. The interval over which the resulting series holds follows from the fact that $\sec x = \infty$ or $\cos x = 0$ for $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

Ex. 3. Show that

$$\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \text{ for } -1 < x < 1.$$

By the binomial theorem

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots, \text{ for } 1 < x < 1.$$

Integrating both sides,

$$\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + c.$$

The value of c, the constant of integration, may be found by noting that $\tan^{-1} 0 = 0$.

Ex. 4. Find approximately the length of a quadrant of the ellipse $x = a \cos \phi$, $y = b \sin \phi$.

$$ds^2 = dx^2 + dy^2 = (a^2 \sin^2 \phi + b^2 \cos^2 \phi) d\phi^2$$
$$= [a^2 - (a^2 - b^2) \cos^2 \phi] \, d\phi^2.$$

Remembering that $\frac{\sqrt{a^2 - b^2}}{a} = e$, the eccentricity of the ellipse,

$$ds = a\sqrt{1 - e^2 \cos^2 \phi} \, d\phi.$$

$$\therefore s = a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \cos^2 \phi} \, d\phi.$$

No expression for the indefinite integral of $\sqrt{1 - e^2 \cos^2 \phi}$ is known. Expanding by the binomial theorem,

$$(1 - e^2 \cos^2 \phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2} e^2 \cos^2 \phi - \frac{1}{2^2 2\,!} e^4 \cos^4 \phi - \frac{1 \cdot 3}{2^3 3\,!} e^6 \cos^6 \phi$$
$$- \frac{1 \cdot 3 \cdot 5}{2^4 4\,!} e^8 \cos^8 \phi - \cdots,$$

which holds for $e^2 \cos^2 \phi < 1$. Since $e < 1$ for an ellipse, and $\cos^2 \phi \overline{<} 1$ for all values of ϕ, this series converges for all values of ϕ. So we can evaluate s by integrating term for term. Doing this, and remembering that, by Wallis' formula (§ 60),

$$\int_0^{\frac{\pi}{2}} \cos^{2n} \phi \, d\phi = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2},$$

$$s = \frac{a\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5} - \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2 \frac{e^8}{7} - \cdots\right]$$

EXERCISES

1. Making use of the fact that $\tan x = \frac{\sin x}{\cos x}$, use the Maclaurin expansions for $\sin x$ and $\cos x$ to find 4 terms of the Maclaurin expansion for $\tan x$.

2. Making use of the fact that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$, obtain the Maclaurin expansion for $\frac{1}{(1-x)^2}$ from the Maclaurin expansion of $\frac{1}{1-x}$ or $(1-x)^{-1}$.

3. Similarly find the expansion for $\frac{1}{(1-x)^3}$.

4. Expand as infinite series

(a) $\int_0^1 \frac{\sin x}{x} dx$; (b) $\int_1^2 \frac{\cos x}{x} dx$; (c) $\int_0^1 e^{-x^2} dx$;

(d) $\int_1^x \frac{e^x}{x} dx$; (e) $\int_0^1 \frac{\log(1+x)}{x} dx$.

150. Theorem of the mean for a function of two variables. If $f(x, y)$ and its derivatives $\frac{\partial f}{\partial x} \equiv f'_x$ and $\frac{\partial f}{\partial y} \equiv f'_y$ are continuous functions in a given region of the plane, in this region

$$f(a+h, b+k) - f(a, b)$$
$$= hf'_x(a+\theta h, b+\theta k) + kf'_y(a+\theta h, b+\theta k). \qquad (1)$$

For, if tentatively we let $x = a + th$ and $y = b + tk$, $f(x, y)$ becomes a function of the single variable t, and we may write

$$f(a+th, b+tk) = F(t).$$

This function and its derivative are continuous, so we can apply the theorem for finite increments (§ 142) to it. Hence

$$F(t+\Delta t) - F(t) = \Delta t\, F'(t+\theta\,\Delta t), \quad \text{where } 0 < \theta < 1.$$

$$\therefore f(a+[t+\Delta t]h,\ b+[t+\Delta t]k) - f(a+th,\ b+tk)$$
$$= \Delta t\, hf'_x(a+[t+\theta\,\Delta t]h,\ b+[t+\theta\,\Delta t]k)$$
$$+ \Delta t\, kf'_y(a+[t+\theta\,\Delta t]h,\ b+[t+\theta\,\Delta t]k). \qquad (2)$$

Noting that $f(a+h,\ b+k) = F(1)$ and $f(a, b) = F(0)$, and that $F(1) - F(0) = F'(\theta)$, we have from (2), when we put $t = 0$ and $\Delta t = 1$,

$$f(a+h,\ b+k) - f(a, b)$$
$$= hf'_x(a+\theta h,\ b+\theta k) + kf'_y(a+\theta h,\ b+\theta k).$$

Thus the theorem is proved.

The theorem can be extended readily to functions of any number of variables.

151. Taylor's theorem for a function of two variables. If $f(x, y)$ and all the derivatives involved are continuous in a certain region of the plane, we can find an expansion for $f(a+h, y+k)$ in terms of positive integral powers of h and k, by proceeding in a manner entirely analogous to that used in the previous section.

Let $F(t) = f(a + th,\ b + tk)$. Developing by Maclaurin's theorem

$$F(t) = F(0) + F'(0)\,t + \frac{F''(0)}{2\,!}t^2 + \cdots + \frac{F^{(n)}(0)}{n\,!}t^n$$
$$+ \frac{F^{(n+1)}(\theta t)}{(n+1)\,!}t^{n+1}. \qquad (1)$$

Since $F'(t) = hf'_x(a + th,\ b + tk) + kf'_y(a + th,\ b + tk)$,

$$F'(0) = hf'_x(a, b) + kf'_y(a, b).$$

Similarly, since

$$F''(t) = h^2 f''_{xx}(a + th,\ b + tk) + 2\,hk f''_{xy}(a + th,\ b + tk)$$
$$+ k^2 f''_{yy}(a + th,\ b + tk),$$
$$F''(0) = h^2 f''_{xx}(a, b) + 2\,hk f''_{xy}(a, b) + k^2 f''_{yy}(a, b).$$

If we write symbolically $(h\mathcal{D}_x + k\mathcal{D}_y)f = hf'_x + kf'_y$, where $\mathcal{D}_x$ stands for $\dfrac{\partial}{\partial x}$ and $\mathcal{D}_y$ stands for $\dfrac{\partial}{\partial y}$, and if we understand

$$(h\mathcal{D}_x + k\mathcal{D}_y)^2 f \text{ to mean } (h^2\mathcal{D}_x^2 + 2\,hk\,\mathcal{D}_x\mathcal{D}_y + k^2\mathcal{D}_y^2)f,$$

that is, $h^2 f''_{xx} + 2\,hk f''_{xy} + k^2 f''_{yy}$; then

$$F'(0) = (h\mathcal{D}_x + k\mathcal{D}_y)f(a, b) \quad \text{and} \quad F''(0) = (h\mathcal{D}_x + k\mathcal{D}_y)^2 f(a, b).$$

Similarly, if $\mathcal{D}_x^m\,\mathcal{D}_y^n$ stands for $\dfrac{\partial^{m+n}}{\partial x^m\,\partial y^n}$, it can be shown step by step that

$$F'''(0) = (h\mathcal{D}_x + k\mathcal{D}_y)^3 f(a, b),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$F^{(n)}(0) = (h\mathcal{D}_x + k\mathcal{D}_y)^n f(a, b),$$
$$F^{(n+1)}(\theta t) = (h\mathcal{D}_x + k\mathcal{D}_y)^{n+1} f(a + \theta\,th,\ b + \theta\,tk).$$

Since $f(a + h, b + k) = F(1)$, we have, on letting $t = 1$ in (1),

$$f(a + h, b + k)$$
$$= f(a, b) + (h\mathcal{D}_x + k\mathcal{D}_y)f(a, b) + \frac{(h\mathcal{D}_x + k\mathcal{D}_y)^2}{2\,!}f(a, b) + \cdots$$
$$+ \frac{(h\mathcal{D}_x + k\mathcal{D}_y)^n}{n\,!}f(a, b) + \frac{(h\mathcal{D}_x + k\mathcal{D}_y)^{n+1}}{(n+1)\,!}f(a + \theta h, y + \theta k). \qquad (2)$$

This is *Taylor's theorem for a function of two variables.* Its extension to functions of any number of variables is obvious.

152. Infinitesimals and infinites of various orders. An infinitesimal, as we have noted before, is a variable which has the limit zero. The infinitesimal need not always be a very small quantity. At some stages of variation it may have large values. The essential thing is that a stage can be named, from and after which the absolute value of the infinitesimal is less than an assigned number no matter how small.

The ratio of two infinitesimals α and β may have a finite limit, including 0, or it may be ∞, as we saw in § 148.

If the limit of the ratio $\frac{\beta}{\alpha}$ is finite and not 0, the infinitesimals are said to be *of the same order;*

if the limit is 0, β is said to be an *infinitesimal of higher order than* α*;*

if the limit is ∞, β is said to be an *infinitesimal of lower order than* α.

If an infinitesimal α is taken as the *standard* or *primary infinitesimal,* a second infinitesimal β, such that the limit of $\frac{\beta}{\alpha^n}$ is finite and different from 0, will be said to be an *infinitesimal of the nth order with respect to* α.

Thus we have seen that since $\underset{\theta \longrightarrow 0}{L} \frac{\sin \theta}{\theta} = 1$, $\sin \theta$ is of the same order of infinitesimals as θ; but since $\underset{\theta \longrightarrow 0}{L} \frac{(1 - \cos \theta)}{\theta^2} = \frac{1}{2}$, $1 - \cos$ is an infinitesimal of the second order with respect to θ.

Similarly if α and β have $\pm \infty$ for limits, they will be known as *infinites.*

If the limit of the ratio $\frac{\beta}{\alpha}$ is finite and different from 0, α and β are said to be *infinites of the same order;*

if the limit is 0, β is said to be an *infinite of lower order than* α*;*

if the limit is ∞, β is said to be an *infinite of higher order than* α.

If an infinite α is taken as the *standard* or *primary infinite*, a second infinite β, such that the limit of $\frac{\beta}{\alpha^n}$ is finite and different from zero, will be said to be an *infinite of the nth order with respect to* α.

Ex. 1. Show that for x an infinite, e^x is an infinite of higher order than x^n where n is any positive integer, no matter how large.

Ex. 2. Show that for x an infinite, $\log x$ is an infinite of lower order than $x^{\frac{1}{n}}$, where n is any positive integer no matter how large.

153. Differentials. While a differential is not necessarily an infinitesimal [it is defined as an increment of the variable of a certain kind (§ 30)] in its applications, it is usually considered as an infinitesimal. This will explain its definition. Thus if

$$y = f(x), \quad \text{then} \quad y + \Delta y = f(x + \Delta x), \text{ and}$$

$$\Delta y = \frac{f(x + \Delta x) - f(x)}{\Delta x} \Delta x.$$

We defined the differential of y as

$$dy = f'(x) \Delta x.$$

$$\therefore \Delta y - dy = \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x)\right] \Delta x, \text{ or}$$

$$\frac{\Delta y - dy}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x). \qquad (1)$$

Now if Δx approaches 0, the right-hand member of (1) approaches 0. Hence, *considering* Δx *as an infinitesimal,* $\Delta y - dy$ *is an infinitesimal of higher order than* Δx. So we see that the differential of a function is made up of that part (the *principal part*) of the increment of the function which is of the first order of infinitesimals with respect to the increment of the independent variable. In the case of the independent variable x, the increment consists of the single term Δx, which is, of course, an infinitesimal of the first order when Δx is the primary infinitesimal. Hence Δx is the differential of the independent variable.

If $y = f(x)$, we have by Taylor's theorem

$$y + \Delta y = f(x + \Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots.$$

$$\therefore \Delta y = f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots. \tag{2}$$

Here we see again that $f'(x)\,\Delta x$ is the part of Δy which is of the first order with respect to Δx.

In chapters XI and XIV various quantities were determined as definite integrals. The differentials of these quantities, which are used in the respective definite integrals, differ from small portions of the determined quantities by infinitesimals of higher order. It is upon this fact that Duhamel's theorem is based.

Formula (2) suggests the definition of higher ordered differentials. Thus we have, as definitions,

the *second differential of* $y \equiv d^2y = y''\,dx^2$,

the *third differential of* $y \equiv d^3y = y'''\,dx^3$,

. . . .

the *nth differential of* $y \equiv d^ny = y^{(n)}\,dx^n$.

Then we can write

$$\Delta y = dy + \frac{d^2y}{2!} + \frac{d^3y}{3!} + \cdots.$$

Obviously *the second and higher differentials of the independent variable are equal to zero.*

With the notion of higher differentials available, the differential notation for higher derivatives naturally suggests itself; thus (§ 31)

$$y'' = \frac{d^2y}{dx^2}, \quad y''' = \frac{d^3y}{dx^2}, \cdots, y^{(n)} = \frac{d^ny}{dx^n}. \tag{3}$$

Attention should be called to the fact, however, that *while* $y' = \dfrac{dy}{dx}$ *holds whether* x *is the independent variable or not, equations* (3) *are true only when* x *is the independent variable.*

Thus it will be left as an exercise to show that, if x and y are functions of a third variable t,

$$y'' = \frac{dx\,d^2y - dy\,d^2x}{dx^3}.$$

From Taylor's theorem for a function of several variables (§ 151), we have, if

$u = f(x, y, z)$, and if we write

$\Delta u = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$, then

$$\Delta u = \frac{\partial f(x, y, z)}{\partial x}\Delta x + \frac{\partial f(x, y, z)}{\partial y}\Delta y + \frac{\partial f(x, y, z)}{\partial z}\Delta z$$
$$+ \text{ terms of higher degree in } \Delta x, \Delta y, \Delta z.$$

Considering Δx, Δy, Δz as infinitesimals of the first order, the part of Δu of the first order is

$$du = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z. \tag{4}$$

This is taken as the *differential of* u (§ 106).

As in the case of a single independent variable, the differential of each of a set of independent variables is the same as its increment; that is,

$$dx = \Delta x, \quad dy = \Delta y, \quad dz = \Delta z. \tag{5}$$

These follow from (4) when $u = x$, y, and z in turn.

154. Differentiation under the integral sign. If $f(x, \alpha)$ is a continuous function of x and α, it can be proved that the integral $\int_{x_1}^{x_2} f(x, \alpha)\, dx$, where x_1 and x_2 are constants or functions of α, is a continuous function of α. Write

$$\phi(\alpha) = \int_{x_1}^{x_2} f(x, \alpha)\, dx. \tag{1}$$

Giving α an increment $\Delta\alpha$, we have

$$\frac{\Delta\phi}{\Delta\alpha} = \frac{1}{\Delta\alpha}\left[\int_{x_1+\Delta x_1}^{x_2+\Delta x_2} f(x, \alpha + \Delta\alpha)\, dx - \int_{x_1}^{x_2} f(x, \alpha)\, dx\right]. \tag{2}$$

Here $x_1(\alpha + \Delta\alpha) = x_1 + \Delta x_1$ and $x_2(\alpha + \Delta\alpha) = x_2 + \Delta x_2$. Using the properties of definite integrals given in § 90, we have

$$\int_{x_1+\Delta x_1}^{x_2+\Delta x_2} f(x, \alpha + \Delta\alpha) dx = \int_{x_1+\Delta x_1}^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{x_2+\Delta x_2} = \int_{x_1}^{x_2} + \int_{x_2}^{x_2+\Delta x_2} - \int_{x_1}^{x_1+\Delta x_1},$$

where the integrand is the same for all the integrals. Using this in (2), we may write, since $\Delta\alpha$ is a constant as far as x is concerned,

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{x_1}^{x_2} \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha}\, dx + \frac{1}{\Delta\alpha}\int_{x_2}^{x_2+\Delta x_2} f(x, \alpha + \Delta\alpha) dx$$
$$- \frac{1}{\Delta\alpha}\int_{x_1}^{x_1+\Delta x_1} f(x, \alpha + \Delta\alpha) dx. \tag{3}$$

Using the theorem for finite increments (3), § 142,

$$f(x, \alpha + \Delta\alpha) - f(x, \alpha) = \Delta\alpha \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\Delta\alpha), \quad 0 < \theta < 1.$$

Using also the theorem of the mean for integration, § 91,

$$\frac{1}{\Delta\alpha}\int_{x_2}^{x_2+\Delta x_2} f(x, \alpha + \Delta\alpha)\, dx = \frac{\Delta x_2}{\Delta\alpha} f(x_2 + \theta_2 \Delta x_2, \alpha + \Delta\alpha), \quad 0 < \theta_2 < 1,$$

and

$$\frac{1}{\Delta\alpha}\int_{x_1}^{x_1+\Delta x_1} f(x, \alpha + \Delta\alpha)\, dx = \frac{\Delta x_1}{\Delta\alpha} f(x_1 + \theta_1 \Delta x_1, \alpha + \Delta\alpha), \quad 0 < \theta_1 < 1.$$

Equation (3) may then be replaced by

$$\frac{\Delta\phi}{\Delta\alpha}=\int_{x_1}^{x_2}\frac{\partial}{\partial\alpha}f(x,\ \alpha+\theta\,\Delta\alpha)\,dx+\frac{\Delta x_2}{\Delta\alpha}f(x_2+\theta_2\,\Delta x_2,\ \alpha+\Delta\alpha)$$
$$-\frac{\Delta x_1}{\Delta\alpha}f(x_1+\theta_1\,\Delta x_1,\ \alpha+\Delta\alpha). \quad (4)$$

Passing to the limit for $\Delta\alpha=0$, we have

$$\frac{d\phi}{d\alpha}=\int_{x_1}^{x_2}\frac{\partial f(x,\ \alpha)}{\partial\alpha}\,dx+\frac{dx_2}{d\alpha}f(x_2,\ \alpha)-\frac{dx_1}{d\alpha}f(x_1,\ \alpha). \quad (5)$$

If x_1 and x_2 are fixed, not depending upon the parameter α,

$$\frac{d\phi}{d\alpha}=\int_{x_1}^{x_2}\frac{\partial f(x,\ \alpha)}{\partial\alpha}\,dx, \quad (6)$$

which says that the derivative of a function of a parameter defined by a definite integral *with fixed limits* may be obtained by differentiating under the integral sign. But when the limits are not fixed, the additional terms given in (5) appear.

APPENDIX

I. HYPERBOLIC FUNCTIONS

1. Hyperbolic functions. Certain combinations of e^x and e^{-x} give rise to functions of x which play a very important rôle. In many respects they resemble the trigonometric functions. As we shall see, their geometric representation ties up with an equilateral hyperbola in a manner analogous to that of the trigonometric functions with a circle. They are called *hyperbolic* functions just as the trigonometric functions are spoken of as *circular* functions. The hyperbolic functions may be defined as follows:

$$\left.\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2}, \text{ read } \textit{hyperbolic sine} \text{ of } x,\\
\cosh x &= \frac{e^x + e^{-x}}{2}, \text{ read } \textit{hyperbolic cosine} \text{ of } x,\\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, \text{ read } \textit{hyperbolic tangent} \text{ of } x,\\
\coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, \text{ read } \textit{hyperbolic cotangent} \text{ of } x,\\
\operatorname{sech} x &= \frac{2}{e^x + e^{-x}}, \text{ read } \textit{hyperbolic secant} \text{ of } x,\\
\operatorname{csch} x &= \frac{2}{e^x - e^{-x}}, \text{ read } \textit{hyperbolic cosecant} \text{ of } x,
\end{aligned}\right\} \quad (1)$$

Other suggested notations for these functions are hysin x, hycos x, hytan x, hycot x, hysec x, hycsc x; also hs x, hc x, ht x, hct x, hsc x, hcsc x.

But the ones we shall use are the generally accepted ones.

From their definitions, it is obvious that

$$\left.\begin{aligned} &\sinh(-x) = -\sinh x,\ \cosh(-x) = \cosh x,\ \tanh(-x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = -\tanh x, \\ &\coth(-x) = -\coth x,\ \operatorname{sech}(-x) = \operatorname{sech} x,\ \operatorname{csch}(-x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = -\operatorname{csch} x, \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} &\tanh x = \frac{\sinh x}{\cosh x},\ \coth x = \frac{\cosh x}{\sinh x},\ \operatorname{sech} x = \frac{1}{\cosh x}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{csch} x = \frac{1}{\sinh x}, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &\cosh^2 x - \sinh^2 x = 1,\ \tanh^2 x + \operatorname{sech}^2 x = 1, \\ &\qquad\qquad\qquad\qquad \coth^2 x - \operatorname{csch}^2 x = 1. \end{aligned}\right\} \tag{4}$$

These formulas are strikingly similar to those connecting the trigonometric functions. As a matter of fact, all the formulas of trigonometry have their analogues, as we shall proceed to illustrate.

From (1) it follows at once that

$$e^x = \cosh x + \sinh x,\ e^{-x} = \cosh x - \sinh x. \tag{5}$$

Using these, along with

$$e^{x+y} = e^x e^y \quad \text{and} \quad e^{-(x+y)} = e^{-x} e^{-y},$$

we have

$$\begin{aligned} &\cosh(x+y) + \sinh(x+y) \\ &\quad = \cosh x \cosh y + \sinh x \sinh y + \cosh x \sinh y + \sinh x \cosh y, \\ &\cosh(x+y) - \sinh(x+y) \\ &\quad = \cosh x \cosh y + \sinh x \sinh y - \cosh x \sinh y - \sinh x \cosh y. \end{aligned}$$

Taking the sum of these, and then the difference, we have

$$\left.\begin{aligned} \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y. \end{aligned}\right\} \tag{6}$$

Replacing y by $-y$, and using (2),

$$\left.\begin{aligned} \cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y, \\ \sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y. \end{aligned}\right\} \tag{7}$$

It will be left as an exercise for the student to show that

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \quad \coth(x \pm y) = \frac{1 \pm \coth x \coth y}{\coth x \pm \coth y}. \tag{8}$$

Letting $y = x$, we have from (6) and (8)

$$\left.\begin{aligned} \sinh 2x &= 2 \sinh x \cosh x, & \tanh 2x &= \frac{2 \tanh x}{1 + \tanh^2 x}, \\ \cosh 2x &= \cosh^2 x + \sinh^2 x, & \coth 2x &= \frac{1 + \coth^2 x}{2 \coth x}. \end{aligned}\right\} \tag{9}$$

Using (4), we have

$$\cosh 2x = 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1; \tag{10}$$

From these, we have

$$\sinh x = \sqrt{\frac{\cosh 2x - 1}{2}}, \quad \cosh x = \sqrt{\frac{\cosh 2x + 1}{2}}, \tag{11}$$

or

$$\sinh \frac{x}{2} = \sqrt{\frac{\cosh x - 1}{2}}, \quad \cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}; \tag{11'}$$

whence

$$\tanh \frac{x}{2} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}, \quad \coth \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{\cosh x - 1}}. \tag{12}$$

From (6) and (7) the following may be obtained by methods entirely analogous to those employed in trigonometry,

$$\left.\begin{aligned} \sinh x + \sinh y &= 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}, \\ \sinh x - \sinh y &= 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}, \\ \cosh x + \cosh y &= 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}, \\ \cosh x - \cosh y &= 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}. \end{aligned}\right\} \tag{13}$$

From the definitions of the hyperbolic functions [or, from these, together with (3)], the student should have little difficulty in obtaining the following differentiation formulas:

XXV. $D_x \sinh u = \cosh u \, D_x u.$

XXVI. $D_x \cosh u = \sinh u \, D_x u.$

XXVII. $D_x \tanh u = \operatorname{sech}^2 u \, D_x u.$

XXVIII. $D_x \coth u = - \operatorname{csch}^2 u \, D_x u.$

XXIX. $D_x \operatorname{sech} u = - \operatorname{sech} u \tanh u \, D_x u.$

XXX. $D_x \operatorname{csch} u = - \operatorname{csch} u \coth u \, D_x u.$

The Maclaurin developments for $\sinh x$ and $\cosh x$ are easily found to be

$$\left.\begin{aligned} \sinh x &= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots, \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \end{aligned}\right\} \tag{14}$$

If we compare these with

$$\begin{aligned} \sin ix &= \frac{ix}{1!} - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \cdots \\ &= i\left[\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right] \\ &= i \sinh x, \\ \cos ix &= 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix^6)}{6!} + \cdots \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ &= \cosh x, \end{aligned}$$

we have the fundamental connections between the hyperbolic and the trigonometric functions

$$\sinh x = - i \sin ix, \quad \cosh x = \cos ix, \tag{15}$$

which will explain the algebraic signs appearing in the formulas of this section.

Formulas (15) could have been taken as the definitions of $\sinh x$ and $\cosh x$; and this is sometimes done. But the fact that the hyperbolic functions are real when the variable is real can be emphasized better by defining them as functions of a real variable than as functions of an imaginary one.

Formulas (15) will also account for the striking resemblance of the first of formulas (5) to Euler's formula (§ 147), and of the formulas (1) defining $\sinh x$ and $\cosh x$ with formulas (6) of § 147.

2. Geometrical representation. Gudermannian. The name, *hyperbolic* functions, was given to the functions under consideration, because geometrical representations for $\sinh u$ and $\cosh u$ with reference to an equilateral hyperbola exist entirely analogous to those of $\sin u$ and $\cos u$ with reference to a circle.

If the equation of the equilateral hyperbola is

$$x^2 - y^2 = a^2,$$

it may be put in the parametric form

$$x = a \cosh u, \quad y = a \sinh u,$$

because of the first of formulas (4).

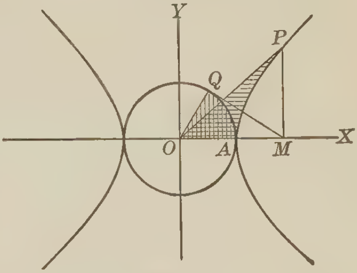

FIG. 1*a*

The area of the sector OPA, which we shall represent by S, is given by $\int_a^x \sqrt{x^2+a^2}\,dx$. Evaluating this, we have

$$S = \tfrac{1}{2} a^2 \log\left(\frac{x}{a} + \frac{y}{a}\right) = \tfrac{1}{2} a^2 \log(\cosh u + \sinh u).$$

Or, using (5), we have finally

$$S = \tfrac{1}{2} a^2 u;$$

whence

$$u = \frac{2}{a^2} S. \tag{16}$$

We have thus shown that *the coördinates of any point P on the equilateral hyperbola $x^2 - y^2 = a^2$ may be represented by*

$$x = a \cosh u, \quad y = a \sinh u \tag{17}$$

where u is the area of the sector OPA multiplied by the constant $\frac{2}{a^2}$, just as the coördinates of any point Q, on the circle $x^2+y^2=a^2$ may be represented by

$$x = a\cos\theta, \quad y = a\sin\theta,$$

where θ[1] *is the area of the sector OQA multiplied by the same constant $\frac{2}{a^2}$.*

Formulas (4), together with (3), suggest the possibility of expressing the hyperbolic functions of u as trigonometric functions of a new variable θ as follows:

$$\left.\begin{array}{lll}\sinh u = \tan\theta, & \cosh u = \sec\theta, & \tanh u = \sin\theta,\\ \coth u = \csc\theta, & \operatorname{sech} u = \cos\theta, & \operatorname{csch} u = \cot\theta.\end{array}\right\} \qquad (18)$$

The geometric representation of this new variable θ can be found readily. In Fig. I*a*, let M be the foot of the ordinate at P, and let Q be the point of contact of the tangent drawn from M to the circle. Then x of the point P, which we saw is equal to $a\cosh u$, is also equal to $a\sec\theta$, since $\sec\theta = \frac{OM}{OQ}$. Hence

$$\cosh u = \sec\theta.$$

The remaining relations of (18) follow at once from this, because of (4) and (3).

The variable θ is known as the *gudermannian*[2] of u, and is written $\theta = gd\,u$. From Fig. I*a* it may be seen readily that, as u varies from $-\infty$ to ∞, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

---

[1] In the case of the circle, θ is usually taken as the angle AOQ. Letting the area of the sector OQA be S', we have $S' = \frac{1}{2}a^2\theta$; whence $\theta = \frac{2}{a^2}S'$.

[2] The name is derived from that of Christof Gudermann (1798–1852), a German mathematician, who played a large part in introducing hyperbolic functions into modern usage.

The graphs of the hyperbolic functions illustrate the relations connecting u and θ given in (16). Thus the graph of $\sinh u$, Fig. I*b*, suggests a single branch of the tangent curve; while that of $\cosh u$, Fig. I*c*, suggests a single branch of the secant curve; and that of $\tanh u$, I*d*, suggests a part of the sine curve.

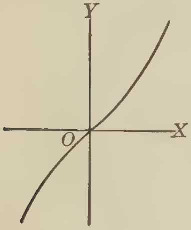

FIG. I*b*

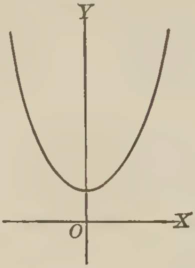

FIG. I*c*

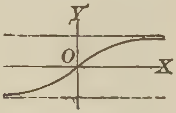

FIG. I*d*

It will be left as an exercise for the student to sketch the graphs of the remaining hyperbolic functions, and to compare them with those of the corresponding trigonometric functions.

3. Inverse hyperbolic functions. Since the hyperbolic functions are expressible in terms of the exponential e^x, it is not surprising that the inverse hyperbolic functions are expressible in terms of $\log x$, which is the inverse function of e^x. Thus if $x = \sinh y$, then $y = \sinh^{-1} x$. Here

$$x = \frac{e^y - e^{-y}}{2}. \quad \therefore e^y - 2x - e^{-y} = 0 \quad \text{or} \quad e^{2y} - 2xe^y - 1 = 0.$$

Solving this quadratic equation in e^y,

$$e^y = x + \sqrt{x^2 + 1};$$

the + sign must be used before the radical because e^y is positive if y is real. Taking the logarithms of both sides,

$$y \equiv \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}). \tag{19}$$

This is real for all real values of x.

In an analogous manner the student should show that

$$\left.\begin{aligned}
\cosh^{-1} x &= \log\left(x \pm \sqrt{x^2-1}\right) \text{ for } x > 1,\\
\tanh^{-1} x &= \tfrac{1}{2}\log\frac{1+x}{1-x} \text{ for } x^2 < 1,\\
\coth^{-1} x &= \tfrac{1}{2}\log\frac{x+1}{x-1} \text{ for } x^2 > 1,\\
\operatorname{sech}^{-1} x &= \log\frac{1 \pm \sqrt{1-x^2}}{x} \text{ for } 0 < x < 1,\\
\operatorname{csch}^{-1} x &= \log\frac{1+\sqrt{1+x^2}}{x}, \text{ for all real values of } x.
\end{aligned}\right\} \quad (20)$$

The student should draw the graph of each of these inverse functions.

From (19) and (20), or directly from the definitions of the inverse hyperbolic functions, we have the differentiation formulas

$$\text{XXXI.} \qquad D_x \sinh^{-1} u = \frac{D_x u}{\sqrt{u^2+1}}.$$

$$\text{XXXII.} \qquad D_x \cosh^{-1} u = \frac{\pm D_x u}{\sqrt{u^2-1}}.$$

$$\text{XXXIII.} \qquad D_x \tanh^{-1} u = \frac{D_x u}{1-u^2}.$$

$$\text{XXXIV.} \qquad D_x \coth^{-1} u = \frac{D_x u}{1-u^2}.$$

$$\text{XXXV.} \qquad D_x \operatorname{sech}^{-1} u = \frac{\pm D_x u}{u\sqrt{1-u^2}}.$$

$$\text{XXXVI.} \qquad D_x \operatorname{csch}^{-1} u = \frac{-D_x u}{u\sqrt{1+u^2}}.$$

These suggest the following integration formulas, the first three being alternatives for the corresponding formulas in § 50.

$$\left.\begin{aligned}
&\int\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\frac{x}{a}+C. \quad \int\frac{dx}{\sqrt{x^2-a^2}}=\pm\cosh^{-1}\frac{x}{a}+C.\\
&\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a}+C \quad \text{or} \quad \frac{1}{a}\coth^{-1}\frac{x}{a}+C.\\
&\int\frac{dx}{x\sqrt{a^2-x^2}}=\pm\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}+C=\pm\frac{1}{a}\cosh^{-1}\frac{a}{x}+C.\\
&\int\frac{dx}{x\sqrt{a^2+x^2}}=-\frac{1}{a}\operatorname{csch}^{-1}\frac{x}{a}+C=-\frac{1}{a}\sinh^{-1}\frac{a}{x}+C.
\end{aligned}\right\}\tag{21}$$

From these we have the following definite integrals, which are real for the indicated values of the variable:

$$\left.\begin{aligned}
&\int_0^x\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\frac{x}{a},\text{ for all real values of } x.\\
&\int_a^x\frac{dx}{\sqrt{x^2-a^2}}=\pm\cosh^{-1}\frac{x}{a},\text{ for } x>a.\\
&\int_0^x\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a},\text{ for } x^2<a^2.\\
&\int_x^\infty\frac{dx}{x^2-a^2}=\frac{1}{a}\coth^{-1}\frac{x}{a},\text{ for } x^2>a^2.
\end{aligned}\right\}\tag{22}$$

II. SOLID ANALYTIC GEOMETRY

1. Rectangular coördinates. Those elements of solid analytic geometry which were presupposed as known in the main body of the book, will be recalled here. It will be well to have in mind the principles and methods of plane analytic geometry and note how these are extended to the consideration of space geometry.

The element in both geometries is the point. The simplest locus in plane geometry is the straight line. In space, the simplest locus or configuration is the plane. The straight line is not so simple, as we shall see.

In plane analytic geometry the position of a point is frequently determined by its rectangular cartesian coördinates, that is, by its distances from two mutually perpendicular lines, known as the axes of coördinates. Correspondingly in space, we can fix the position of a point by determining its distances from three mutually perpendicular planes, known as the *coördinate planes.* Thus in Fig. IIa, the planes YOZ, XOZ, XOY are supposed to be mutually perpendicular.

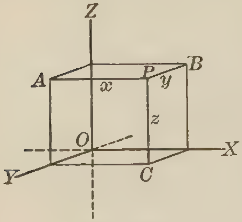

FIG. IIa

If P is a point in space, PA, PB, PC, its respective distances from these planes, are taken as its coördinates, and, analogously to the notation in plane analytic geometry, are designated by x, y, z, respectively.

A convenient modification, which enables one to make a simpler diagram, is to bring into evidence OX, OY, OZ, the lines of intersection of the coördinate planes and use these as coördinate axes in the following manner: Dropping a perpendicular from P upon the plane XOY we obtain C. This is said to be the projection of P upon the plane. Similarly, drop a perpendicular from C upon OX, obtaining M, its projection upon the line. Then evidently

$$x = OM,\ y = MC,\ z = CP.$$

Since x is measured in the direction of OX, the latter is known as the *x-axis*. Correspondingly, OY and OZ are known as the *y-axis* and *z-axis* respectively.

Similarly, since each of the coördinate planes contains two of the coördinate axes, the plane YOZ is known as the *yz-plane*, the plane XOZ as the *xz-plane*, and the plane XOY as the *xy-plane*.

The point O, which is the common intersection of the coördinate planes and also of the coördinate axes, is known as the *origin*.

The coördinate planes divide all space into eight compartments or octants. Points in the various compartments are distinguished by the algebraic signs of their coördinates analogously to what is done in plane analytic geometry. Positive and negative directions are chosen along each axis, and each coördinate has the positive or negative sign according as it is measured in the positive or negative direction of the corresponding coördinate axis.

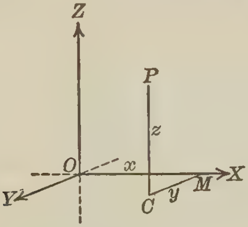

Fig. II*b*

In Fig. II*b*, the positive directions along the axes are indicated by arrow heads. Then the coördinates of P, as drawn, are all positive.

2. Distance between two points. Let P_1 and P_2 be two given points, whose coördinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Let their projections on the xy-plane be C_1 and C_2. In Fig. IIc,

$$x_2 - x_1 = OM_2 - OM_1 \\ = M_1M_2 = C_1D.$$

$$y_2 - y_1 = M_2C_2 - M_1C_1 \\ = DC_2.$$

$$z_2 - z_1 = C_2P_2 - C_1P_1 \\ = EP_2.$$

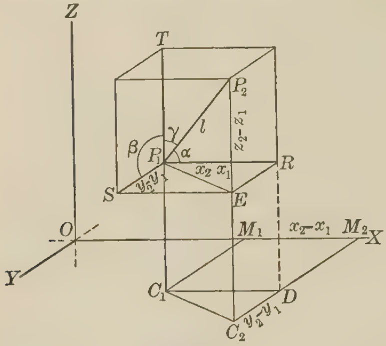

FIG. IIc

In the upper part of the figure we have a rectangular parallelopiped with P_1 and P_2 a pair of opposite vertices, and each of the edges parallel to a coördinate axis and equal to the difference between the corresponding coördinates of P_1 and P_2. Thus

$$P_1R = x_2 - x_1, \quad P_1S = y_2 - y_1, \quad P_1T = z_2 - z_1.$$

Since $\overline{P_1P_2}^2 = \overline{P_1E}^2 + \overline{EP_2}^2 = \overline{P_1E}^2 + (z_2 - z_1)^2$, and

$$\overline{P_1E}^2 = \overline{P_1R}^2 + \overline{RE}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Letting $P_1P_2 = l$, we have the important formula

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad (1)$$

This formula is entirely analogous to the corresponding one for two dimensions.

In general, if we let all the z-coördinates which appear in any formula be zero, the resulting formula must hold for two dimensions.

3. Direction cosines. In plane analytic geometry the slope of a line, that is the tangent of the angle which the line makes with the x-axis arises as the simplest means of determining the direction of the line. In solid analytic geometry it is found convenient to use all three angles which a line makes with the coördinate axes[1] in determining its direction in space.

These angles are known as the *direction angles* of the line, the direction angles with respect to the x-, y-, z-axis being usually denoted by α, β, γ respectively.

In order to fix these angles precisely, we assume one or the other direction along the line as the positive direction along it, and then say that the direction angles of the line are the angles which this positive direction makes with the positive directions of the respective coördinate axes. By a proper choice of the positive direction along a line, at least two of the direction angles can always be made acute.

The cosines of these angles are easily expressed in terms of the coördinates of two points on the line, and are known as the *direction cosines* of the line. Thus in Fig. IIc

$$\left.\begin{aligned} \cos\alpha &= \frac{P_1R}{P_1P_2} = \frac{(x_2 - x_1)}{l}, \\ \cos\beta &= \frac{P_1S}{P_1P_2} = \frac{(y_2 - y_1)}{l}, \\ \cos\gamma &= \frac{P_1T}{P_1P_2} = \frac{(z_2 - z_1)}{l}. \end{aligned}\right\} \qquad (2)$$

These formulas are deduced on the supposition that the line P_1P_2 is directed positively from P_1 toward P_2. The statement that, at least two of the direction angles, can always be made acute by a proper choice of the positive direction along the line is equivalent to saying that, at least two of the direction cosines can always be made positive. From (2) we see that this is substantiated if the positive direction along the line P_1P_2 is taken in such a way that at least two of the differences of corresponding coördinates appearing in (2) are positive. Here l is taken as positive.

---

[1] It is understood that the angle which a line makes with a non-intersecting line in space is the angle which it makes with an intersecting line parallel to the other line.

The angles α, β, γ are evidently not independent. The analytic relation between them is

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \tag{3}$$

as is obvious on squaring both members of each of equations (2) and adding, making use of (1).

If, in the plane, instead of using a single angle to determine the direction of a line, the two angles α and β which the line makes with the axes of x and y respectively are employed

$$\beta = \pm\left(\frac{\pi}{2} - \alpha\right) \text{ according as } \alpha \text{ is acute or obtuse.}$$

$$\therefore \cos\beta = \sin\alpha, \text{ and } \cos^2\alpha + \cos^2\beta = 1.$$

Remark. — In practice, when two points on a line are known, use is made of the fact that the direction cosines of the line are proportional to the differences of the corresponding coördinates; that is

$$\cos\alpha : \cos\beta : \cos\gamma = x_2 - x_1 : y_2 - y_1 : z_2 - z_1.$$

This follows at once from (2). To find the direction cosines, we divide each of these differences by the square root of the sum of their squares.

Thus for the line adjoining (6, 3, − 4) and (4, 0, 2)

$$\cos\alpha : \cos\beta : \cos\gamma = -2 : -3 : 6 = 2 : 3 : -6.$$

$$\therefore \cos\alpha = \tfrac{2}{7}, \quad \cos\beta = \tfrac{3}{7}, \quad \cos\gamma = -\tfrac{6}{7}.$$

Here the positive direction along the line must be taken from (4, 0, 2) toward (6, 3, − 4) if two of the direction cosines are to be positive.

EXERCISES

Find the distances between the following points and the direction cosines of the angles of the lines joining them.

1. (0, 0, 0) and (2, − 2, 2).
2. (3, 1, 4) and (1, − 3, −1).
3. (4, 2, − 1) and (5, − 1, 1).
4. (1, 2, 3) and (3, 0, 5).
5. (5, − 1, 2) and (6, 1, 0).
6. (2, 3, 0) and (1, 4, 2).

4. Angle between two lines. If GH and KL (Fig. IId) are any two lines in space, whether intersecting or not, the angle between them is defined as the angle between two similarly directed and intersecting lines, one of which is parallel to GH and the other parallel to KL.

Let OP_1 and OP_2 be two such lines through the origin. Given their direction angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ respectively, let θ be the angle between the lines. We can find the value of $\cos\theta$ in terms of the direction cosines of the lines.

We shall first have to introduce the notion of the projection of a line-segment upon another line. We shall be dealing here with *orthogonal* projection only.

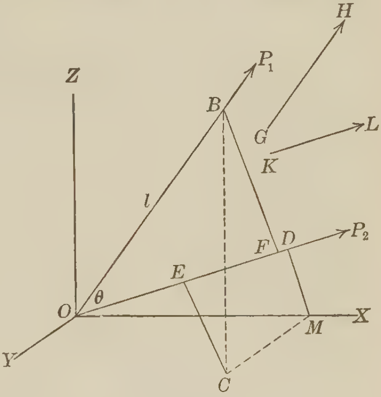

FIG. II*d*

The foot of the perpendicular dropped from a point on a line, is defined as the *projection of the point* upon the line. In Fig. II*d* the lines CE, BF, MD are supposed to be perpendicular to OP_2. (They do not appear to be so because the lines do not all lie in the plane of the drawing). Then E, F, D are the projections of C, B, M respectively upon OP_2.

By the *projection of a line-segment* upon another line is meant the portion of the second line lying between the projections of the extremities of the line-segment upon that line. Thus, in the figure OF, ED, EF, and OD are the projections of OB, CM, CB, and OM respectively upon OP_2. If the line-segment is directed, its projection will be also. Thus, taking into account its direction, the projection of OB upon OP_2 is OF while that of BO is FO.

Instead of going from O to B along the straight line, we can first go from O to M along OM, thence to C along MC, and finally to B along CB. Here the line-segments are directed. We have seen the projections of these upon OP_2 are OD, DE,

EF respectively, and their sum is OF, the projection of OB upon OP_2. This is illustrative of the general theorem (which is not difficult to prove) that *the algebraic sum of the projections upon a line L of the segments constituting a broken line is equal to the projection upon the line L of the line joining the extremities of the broken line.*

From its definition, it is easy to see that *the projection of a line-segment upon a line is equal to the length of the line-segment multiplied by the cosine of the angle between the two lines.*

Thus $OF = l \cos\theta$, where l is the length of OB. Also, if the coördinates of B are (x_1, y_1, z_1),

$$OD = x_1 \cos\alpha_2, \quad DE = y_1 \cos\beta_2, \quad EF = z_1 \cos\gamma_2.$$

Hence using the theorem established above

$$l \cos\theta = x_1 \cos\alpha_2 + y_1 \cos\beta_2 + z_1 \cos\gamma_2,$$

or

$$\cos\theta = \frac{x_1}{l}\cos\alpha_2 + \frac{y_1}{l}\cos\beta_2 + \frac{z_1}{l}\cos\gamma_2.$$

Finally, using (2) we have the important formula

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \tag{4}$$

Frequently, instead of using the direction cosines themselves we shall find it convenient to deal with quantities proportional to them, such as the differences between the corresponding coördinates of two points on a line. If the direction cosines of the two lines are proportional to $l_1 : m_1 : n_1$ and $l_2 : m_2 : n_2$ respectively, it will be left as a simple exercise to show that

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}. \tag{4'}$$

From (4) and (4′) we find the condition that the two lines be mutually perpendicular is

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0, \tag{5}$$

or

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0, \tag{5'}$$

since in this case $\cos\theta = 0$.

What information do (4) and (4′) give when the two lines are parallel?

EXERCISES

Find the cosines of the angles between the various lines in the exercises of § 3. Which of these are mutually perpendicular? Which are parallel?

5. Coördinates of any point on line through two given points. In Fig. II*e*, P_1 and P_2 are two points whose coördinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let us find (x, y, z), the coördinates of the point P whose distances from P_1 and P_2 respectively are in the ratio of $\frac{m}{n}$; *i.e.*

$$\frac{P_1P}{PP_2}=\frac{m}{n} \quad \text{or} \quad \frac{P_1P}{P_1P_2}=\frac{m}{m+n}.$$

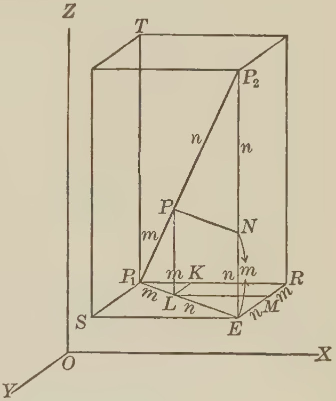

FIG. II*e*

In Fig. II*e*, we have the parallelopiped having P_1 and P_2 for a pair of opposite vertices and its edges all parallel to one or another of the coördinate axes. In addition

P_1E is the projection of P_1P_2 upon the plane of P_1R and P_1S,
L is the projection of P upon this plane,
K is the projection of L upon P_1R,
M is the projection of L upon RE,
N is the projection of P upon EP_2.

Then by similar triangles

$$\frac{P_1K}{P_1R}=\frac{P_1L}{P_1E}=\frac{P_1P}{P_1P_2}=\frac{m}{m+n}. \tag{6}$$

Noting that $P_1K=x-x_1$ and $P_1R=x_2-x_1$, we have, on putting these in (6) and solving for x,

$$x=\frac{mx_2+nx_1}{m+n}. \tag{7}$$

It will be left as an exercise to show that

$$y=\frac{my_2+ny_1}{m+n},\quad z=\frac{mz_2+nz_1}{m+n}. \tag{7'}$$

For the coördinates of the middle point ($m=n$)

$$x=\frac{x_1+x_2}{2},\quad y=\frac{y_1+y_2}{2},\quad z=\frac{z_1+z_2}{2}. \tag{8}$$

As in the case of plane analytic geometry, the formulas (7) and (7′) determine points on the line external to the line-segment P_1P_2 when $\frac{m}{n}$ is negative.

6. Equation of a surface; of a curve. A single relation between the coördinates of a variable point in space

$$F(x, y, z)=0 \tag{9}$$

defines a surface. Suppose (9) solved for one of the variables, say z, giving

$$z=f(x, y). \tag{10}$$

We may choose x and y arbitrarily. Then z is determined either uniquely or as one of a set of values. If at each point (x, y) in the xy-plane, we imagine a perpendicular erected, in length equal to the value or values of z given by (10), the aggregate of these extremities of perpendiculars will be the surface determined by (9) or (10).

In space, then, a single relation between x, y, z defines a surface, in which we have two degrees of freedom (we can select two of the coördinates at will) while in the plane, a single relation between x and y defines a curve, in which we have one degree of freedom (we can select only one of the coördinates at will).

A more symmetrical way of defining a surface analytically is to have the coördinates of each point on it given in terms of two parameters u and v, thus

$$x=f(u, v),\quad y=\phi(u, v),\quad z=\psi(u, v). \tag{11}$$

For example

$$x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = a \sin u \tag{12}$$

determines the points on a sphere of radius a with its center at the origin; u and v being the latitude and longitude respectively of the point. To obtain the rectangular equation of the surface, u and v must be eliminated between (12). This can be done readily in this case by squaring both members of each of the equations (12) and adding, thus obtaining

$$x^2 + y^2 + z^2 = a^2. \tag{13}$$

That (13) represents a sphere of radius a with its center at the origin is obvious from the fact that its left-hand member is the square of the distance of any point (x, y, z) on the surface from the origin.

Two relations

$$F_1(x, y, z) = 0 \quad \text{and} \quad F_2(x, y, z) = 0 \tag{14}$$

define a curve, which is the intersection of the two surfaces defined by each of the relations separately. Solving (14) for two of the variables in terms of the third, say

$$y = f_1(x), \quad z = f_2(x), \tag{15}$$

we see that one of the variables only can be selected at will, the other two being determined in terms of it.

In this case also a more symmetrical way of defining a curve analytically is to have the coördinates of each point on it given in terms of a single parameter.

$$x = f(u), \quad y = \phi(u), \quad z = \psi(u). \tag{16}$$

7. Equation of the first degree. Given the equation

$$ax + by + cz + d = 0, \tag{17}$$

if (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two points on the surface determined by (17),

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{and} \quad ax_2 + by_2 + cz_2 + d = 0. \tag{18}$$

Multiplying these, respectively, by n and m, any two numbers, and adding, we have

$$a(nx_1 + mx_2) + b(ny_1 + my_2) + c(nz_1 + mz_2) + d(n + m) = 0,$$

or

$$a\frac{nx_1 + mx_2}{m + n} + b\frac{ny_1 + my_2}{m + n} + c\frac{nz_1 + mz_2}{m + n} + d = 0. \qquad (19)$$

Recalling (7) and (7′), equation (19) informs us that if the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the surface (17), so also does $\left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}, \frac{nz_1 + mz_2}{m + n}\right)$, for all values of m and n. That is, every point on the line joining the two points is on the surface. But this is the definition of a plane. Hence we have proved that *an equation of the first degree represents a plane.*

In particular $x = a$, where a is a constant, is the plane parallel to the yz-plane and at a distance a from it; $x = 0$ is the yz-plane itself. Similarly $y = b$, where b is a constant, is the plane parallel to the xz-plane and at a distance b from it; $y = 0$ is the xz-plane itself. And $z = c$, where c is a constant, is the plane parallel to the xy-plane and at a distance c from it; $z = 0$ is the xy-plane itself.

If (x_1, y_1, z_1) is a fixed point on the plane (17), its coördinates satisfy the equation (17); that is

$$ax_1 + by_1 + cz_1 + d = 0.$$

Subtracting this from (17), we have

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \qquad (20)$$

Here (x, y, z) is any point on the plane. Hence (20) may be taken as the equation of the plane. In this form of the equation, the fact that the plane contains the point (x_1, y_1, z_1) is brought into evidence.

Moreover a, b, c have a geometrical significance. The differences $x - x_1$, $y - y_1$, $z - z_1$ are proportional to the direction cosines of the line joining (x, y, z) with (x_1, y_1, z_1). Equation (20) says that this line is perpendicular to a line l, whose direction cosines are proportional to a, b, c. This line l is thus perpendicular to every line of the plane, through (x_1, y_1, z_1). It is said to be a *normal* to the plane. We thus see that *the coefficients* a, b, c *in the equation* (17), *or* (20), *of the plane are proportional to the direction cosines of a normal to the plane.* Evidently all normals to a plane are parallel and thus have the same direction cosines. Equation (20) is a very useful form of the equation of a plane. It can be written at once when a point on the plane and the direction cosines of a normal to the plane are known.

Two equations of the first degree

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0 \tag{21}$$

represent a straight line, which is the intersection of the two planes represented by the equations (21), each taken separately.

A serviceable form of the equations of a straight line is obtained when use is made of the fact that a straight line is determined by a point on it and its direction. Let (x_1, y_1, z_1) be the point on the line and let l, m, n be quantities proportional to the direction cosines of the line. If (x, y, z) is any other point on the line, $x - x_1$, $y - y_1$, $z - z_1$, are proportional to its direction cosines. Hence we must have

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \tag{22}$$

These are the equations of the straight line through the point (x_1, y_1, z_1) and having its direction cosines proportional to l, m, n.

Thus, for example,

$$3(x - 1) + 2y - (z + 2) = 0$$

is the equation of a plane, through $(1, 0, -2)$, whose normals have direction cosines proportional to $3, 2, -1$. And

$$\frac{x - 1}{3} = \frac{y}{2} = \frac{z + 2}{-1}$$

are the equations of the normal to the plane through the point $(1, 0, -2)$.

8. Cylinder. If one of the variables is absent from an equation, that variable is undetermined. Thus if we have

$$F(x, y) = 0, \tag{23}$$

z can take any value. In particular, if we put $z = 0$, this, together with (23), determines a curve in the xy-plane. If at some point of this curve a perpendicular to the xy-plane is erected, the coördinates of any point on this line will satisfy equation (23). Hence the whole line lies on the surface. This being true for all the lines perpendicular to the xy-plane and passing through some point of the curve in that plane we see that (23) determines a cylinder with this curve as directrix and with its elements parallel to the z-axis.

Similarly the equations

$$F(y, z) = 0 \quad \text{and} \quad F(x, z) = 0$$

determine cylinders with their elements parallel to the x- and y-axes respectively. Thus

$$x^2 + y^2 = 25, \quad \frac{y^2}{9} + \frac{z^2}{4} = 1, \quad z^2 = 4x$$

represent a right circular cylinder, an elliptic cylinder, and a parabolic cylinder with their elements parallel to the z-axis, the x-axis, and the y-axis respectively.

More generally, if $F(x, y, z)$ can be written as a function of two linear functions of the variables $a_1x_1 + b_1y + c_1z + d_1$ and $a_2x + b_2y + c_2z + d_2$, $F(x, y, z) = 0$ represents a cylinder. For making a transformation of coördinates

$$X = a_1x + b_1y + c_1z + d_1, \quad Y = a_2x + b_2y + c_2z + d_2, \quad Z = z,$$

the resulting equation of the surface is of the form $\Phi(X, Y) = 0$, which involves two of the coördinates only. Even if the new cartesian coördinates are not rectangular, the argument employed above applies here.

Also $3x + 4z - 12 = 0$ is a cylinder with its elements parallel to the y-axis and the line $3x + 4z - 12 = 0$, $y = 0$ in the xz-plane as directrix. But as a cylinder with a line for a directrix is a plane, it is simpler to consider this as a plane. The direction cosines of a normal are proportional to 3, 0, 4.

If one only of the variables, say z, appears in the equation, it has the form

$$F(z)=0. \tag{24}$$

This may be supposed solved for z, giving rise to a set of n equations, if $F(z)$ is of the n-th degree,

$$z=c_1, \quad z=c_2, \cdots, z=c_n.$$

Each of these is a plane parallel to the xy-plane.

9. Cone. If the equation of a surface is homogeneous in the variables, we shall show that the surface represented by it is a cone with its vertex at the origin. To illustrate, let us consider

$$x^3+y^3-3x^2y+6xz^2-12xyz=0. \tag{25}$$

This is homogeneous because each term is of the third degree.

It evidently passes through the origin, because there is no constant term.

If a point P_1, whose coördinates are (x_1, y_1, z_1) is on the surface

$$x_1^3+y_1^3-3x_1^2y_1+6x_1z_1^2-12x_1y_1z_1=0. \tag{26}$$

Now the coördinates of any other point P on the line joining P_1 with the origin will be of the form $(\lambda x_1, \lambda y_1, \lambda z_1)$, where λ is some constant. Since each term of (25) is of the third degree, the result of substituting the coördinates of P in the left-hand member of (25) is the left-hand member of (26) multiplied by λ^3. Hence the coördinates of P also satisfy the equation (25); that is, P lies on the surface. Since P is any point on the line joining P_1 with the origin, this line lies on the surface. The surface is thus made up of lines through the origin.

The argument used in the special case may now be employed in the general case. For if

$$F(x, y, z)=0 \tag{27}$$

is homogeneous in x, y, z and of degree n (§ 113)

$$F(\lambda x, \lambda y, \lambda z)=\lambda^n F(x, y, z). \tag{28}$$

This is an identity, and holds for any choice of the common factor λ. Hence if (x_1, y_1, z_1) is on the surface, then $(\lambda x_1, \lambda y_1, \lambda z_1)$ is also. That is, the surface contains every point of the line joining (x_1, y_1, z_1) with the origin. But this is the definition of a cone with its vertex at the origin.

In particular if $F(x, y, z)$ is linear in x, y, z (27) still represents a cone with a line for directrix. It is simpler to classify it as a plane through the origin.

Similarly if one of the variables does not appear in (27), the surface represented by (27) is both a cone and a cylinder. But there is a simpler interpretation in this case. Suppose, to fix the idea that x is absent, the equation of the surface being

$$F(y, z) = 0. \tag{29}$$

The identity (28) now takes the form

$$F(\lambda y, \lambda z) \equiv \lambda^n F(y, z). \tag{30}$$

In particular let $\lambda = \frac{1}{z}$; then (30) becomes

$$F(y, z) \equiv z^n F\left(\frac{y}{z}, 1\right),$$

and the equation of the surface may be put into the form

$$F\left(\frac{y}{z}, 1\right) = 0.$$

Solving this for $\frac{y}{z}$, we have

$$\frac{y}{z} = a_1, \quad \frac{y}{z} = a_2, \cdots, \frac{y}{z} = a_n.$$

Each of these is a plane throughout the x-axis. This is a much simpler description of the surface represented by (29).

10. Surface of revolution. If in the equation of a surface two of the variables, say x and y to fix the idea, appear in such a way that $F(x, y, z)$ is actually a function of $x^2 + y^2$ and z, the equation of the surface may be put into the form

$$\Phi(x^2 + y^2, z) = 0. \tag{31}$$

Solving for $x^2 + y^2$,

$$x^2 + y^2 = \phi_1(z), \quad x^2 + y^2 = \phi_2(z), \cdots, x^2 + y^2 = \phi_n(z), \quad (32)$$

where n is the degree to which $x^2 + y^2$ appears in (31).

The plane $z = a$, cuts the surface in the circles

$$\left\{\begin{matrix} x^2 + y^2 = \phi_1(a) \\ z = a \end{matrix}\right\}, \quad \left\{\begin{matrix} x^2 + y^2 = \phi_2(a) \\ z = a \end{matrix}\right\}, \cdots, \left\{\begin{matrix} x^2 + y^2 = \phi_n(a) \\ z = a \end{matrix}\right\}.$$

These circles have their centers at $(0, 0, a)$, which is on the z-axis.

This holds for all values of a; that is, any plane perpendicular to the z-axis cuts the surface in one or more circles with their centers in the z-axis. A little reflection will show that the surface may be considered as generated by the revolution of either the plane curve $\Phi(x^2, z) = 0$, $y = 0$, or the plane curve $\Phi(y^2, z) = 0$, $x = 0$, about the z-axis, and is known as a *surface of revolution.*

Similarly, if the equation can be put into either of the forms

$$\Phi(y^2 + z^2, x) = 0 \quad \text{or} \quad \Phi(x^2 + z^2, y) = 0, \qquad (31')$$

the surface is one of revolution about the x- or y-axis, respectively.

The introduction of the cylindrical coördinates (ρ, θ, z), § 125, reduces (31), the equation of the surface, to the simple form $\Phi(\rho^2, z) = 0$, involving two variables only, and suggesting the equation of the curve generating the surface. Similarly, the introduction of each set of cylindrical coördinates (ρ, θ, x) and (ρ, θ, y) reduces the corresponding equations of (31′) to a simpler form, each involving two variables only.

11. Special surfaces. Given an equation

$$F(x, y, z) = 0, \qquad (33)$$

it is not always easy to classify the surface represented by it. In the preceding sections we have learned that

if $F(x, y, z)$ is of the first degree, (33) is the equation of a plane;

if one of the variables does not appear in $F(x, y, z)$, (33) is the equation of a cylinder, whose elements are parallel to the corresponding coördinate axis;

if one only of the variables appears in $F(x, y, z)$, (33) is the equation of one or more planes all parallel to the corresponding coördinate plane;

if $F(x, y, z)$ is homogeneous in the three variables, (33) is the equation of a cone with its vertex at the origin;

if $F(x, y, z)$ is homogeneous in two of the variables, the third one being absent, (33) is the equation of a set of planes through the corresponding coördinate axis;

if the introduction of the proper cylindrical coördinates reduces the equation to one of the forms

$$\Phi(\rho^2, x) = 0, \quad \Phi(\rho^2, y) = 0, \quad \Phi(\rho^2, z) = 0,$$

the surface is one of revolution about the corresponding coördinate axis.

We often learn something of the nature of the surface by noting whether it is symmetrical with respect to the origin or any of the coördinate planes or axes, whether it extends indefinitely in various directions or is restricted, by studying the kind of sections made with it by various planes, and such other more or less obvious devices as may suggest themselves in any particular case.

The following special types of surfaces are worth noting:

Ellipsoid. The surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{34}$$

is known as an *ellipsoid.*

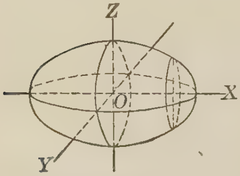

FIG. II*f*

The plane $x = x_0$ cuts it in the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x_0^2}{a^2}, \quad x = x_0.$$

This ellipse is real provided $x_0^2 \overline{<} a^2$, that is $-a \leq x_0 < a$.

Similarly the plane $y = y_0$ cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{y_0^2}{b^2}, \quad y = y_0,$$

which is real provided $-b \leq y_0 \leq b$.

And the plane $z = z_0$ cuts the surface in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_0^2}{c^2}$, $z = z_0$, which is real, provided $-c \leq z_0 \leq c$.

Moreover, it can be proved that every plane cuts the surface in an ellipse. Hence its name.

The surface lies entirely in the rectangular parallelopiped whose faces are $x = a$, $x = -a$, $y = b$, $y = -b$, $z = c$, $z = -c$.

The coördinate planes are planes of symmetry of the surface, the axes of coördinates are axes of symmetry of the surface, and the origin is the center of symmetry or simply "the center" of the surface.

If $a = b = c$, the ellipsoid becomes a sphere.

If $a = b$, $a = c$, or $b = c$, the surface is an ellipsoid of revolution.

The surface

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1 \tag{35}$$

is an ellipsoid whose center is the point (h, k, l). The planes $x = h$, $y = k$, $z = l$ are planes of symmetry, and the lines

$$\begin{Bmatrix} y = k \\ z = l \end{Bmatrix}, \quad \begin{Bmatrix} z = l \\ x = h \end{Bmatrix}, \quad \begin{Bmatrix} x = h, \\ y = k \end{Bmatrix}$$

are axes of symmetry.

Hyperboloid of one sheet. The surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{36}$$

is known as an *hyperboloid of one sheet.*

The plane $x = x_0$ cuts it in the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{x_0^2}{a^2}, \quad x = x_0.$$

This is real for all values of x_0.

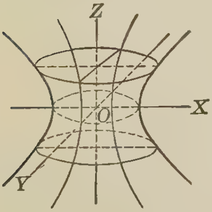

FIG. II*g*

The plane $y = y_0$ cuts the surface in the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y_0^2}{b^2}, \quad y = y_0,$$

which is real for all values of y_0.

The plane $z = z_0$ cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z_0^2}{c^2}, \quad z = z_0,$$

which is real for all values of z_0.

It can be proved that any plane cuts the surface in a real ellipse or a real hyperbola. (In the special case when the plane is tangent to the surface, the hyperbola of intersection is a pair of intersecting lines.) The surface is said to be "of one sheet" because it is made up of one part; starting at any point of the surface any other point of the surface can be reached by moving along the surface. The significance of this will be appreciated when we come to consider the hyperboloid of two sheets.

The coördinate planes are planes of symmetry for the surface, the coördinate axes are axes of symmetry, and the origin is the center of the surface.

If $a = b$, the surface is one of revolution, obtained by revolving the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$, $y = 0$ about its conjugate axis. The surface

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1 \tag{38}$$

also is an hyperboloid of one sheet. Where are its center, planes and axes of symmetry?

Hyperboloid of two sheets. The surface whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{39}$$

is known as *an hyperboloid of two sheets.*

The plane $x = x_0$ cuts it in the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x_0^2}{a^2} - 1, \quad x = x_0.$$

This is real if $x_0 \geqq a$ or $x_0 \leqq -a$. That is, there is no part of the surface between the planes $x = -a$ and $x = a$.

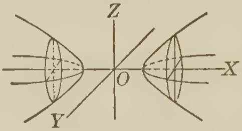

FIG. IIh

The planes $y = y_0$ and $z = z_0$ cut the surface in real hyperbolas for all values of y_0 and z_0.

It can be proved that any plane cuts the surface in a real hyperbola or in a real or imaginary ellipse. The surface is made up of two parts, one lying to the left of the plane $x = -a$ and the other to the right of the plane $x = a$. Hence it is said to be "of two sheets."

The coördinate planes are planes of symmetry, the coördinate axes are axes of symmetry, and the origin is the center of the surface.

If $b = c$, the surface is one of revolution, obtained by revolving the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$, $y = 0$ about its transverse axis.

What can be said of the surface

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1\,? \tag{40}$$

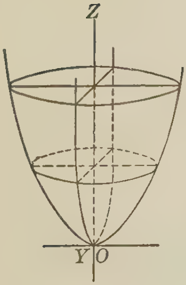

FIG. IIi

Elliptic paraboloid. The surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\,cz \tag{41}$$

is known as an *elliptic paraboloid.*

The planes $x = x_0$ and $y = y_0$ cut it in real parabolas for all values of x_0 and y_0.

The plane $z = z_0$ cuts it in a real ellipse if $z_0 \geqq 0$. The surface lies entirely above the xy-plane.

It can be shown that any plane cuts the surface in either a parabola or an ellipse.

The xz- and yz-planes are planes of symmetry, and the z-axis is an axis of symmetry. The surface has no center.

If $a = b$, the surface is one of revolution, obtained by revolving the parabola $\frac{x^2}{a^2} = 2\,cz$, $y = 0$ about its axis.

What can be said of the surface

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 2\,c(z-l)\,? \tag{42}$$

Hyberbolic paraboloid. The surface whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,cz$$

is known as an *hyperbolic paraboloid.*

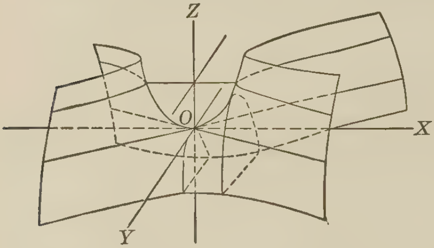

FIG. IIj

The planes $x = x_0$ and $y = y_0$ cut it in real parabolas, for all values of x_0 and y_0; in the former case the parabolas lie above the xy-plane, in the latter case they lie below that plane.

The plane $z = z_0$ cuts the surface in a real hyperbola for every value of z_0; for $z_0 = 0$ the hyberbola breaks up into the pair of lines $\frac{x}{a} - \frac{y}{b} = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$, $z = 0$.

There are other lines in the surface. Since the equation of the surface may be written

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 2\,cz, \tag{43}$$

each of the planes $\frac{x}{a} - \frac{y}{b} = c_1$ and $\frac{x}{a} + \frac{y}{b} = c_2$, for c_1 and c_2 any constants, cuts the surface in a line.

It can be shown that any plane cuts the surface in either an hyperbola, a parabola, a pair of lines, or a single line.

The xz- and yz-planes are planes of symmetry, and the z-axis is an axis of symmetry.

If $a=b$, the hyperbolas cut out by planes parallel to the xy-plane are equilateral. Taking the vertical planes through the common asymptotes of these equilateral hyperbolas as coördinate planes, the equation of the surface assumes the form

$$xy=2\,az. \tag{44}$$

What is the nature of the various types of section of this surface by planes parallel to each of the coördinate planes?

What can be said of the surfaces

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=2\,c(z-l), \tag{45}$$

and

$$(x-h)(y-k)=2\,a(z-l)? \tag{46}$$

Remark. — If its equation $F(x, y, z)=0$ is of the second degree, the surface is frequently spoken of as a *quadric surface.* The types of quadric surfaces enumerated above are the ellipsoid (including the sphere), the hyperboloid of one and two sheets, and the elliptic and hyperbolic paraboloids. In addition, there are the quadric cylinder and the quadric cone, in each of which the directrix is a conic. According as the directrix is an ellipse, parabola, or hyperbola, the cylinder will be said to be *elliptic*, *parabolic*, or *hyperbolic.* These types are essentially distinct. But the matter is different in the case of cones. Since from the same quadric cone can be obtained sections which are ellipses (including circles), parabolas, and hyperbolas, depending upon the direction of the cutting plane, it is clear that the same quadric cone may have either an ellipse, a parabola, or an hyperbola for directrix. All quadric cones are not superposable, one upon the other. Some may be surfaces of revolution, in which case a section normal to the axis of the cone is a circle. Others may have this section an ellipse of any eccentricity between 0 and 1. But there is no distinction between an elliptic cone, a parabolic cone, and an hyperbolic cone.

It should be added that if $F(x, y, z)$ is the product of two linear factors $(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2)$, the quadric surface consists of the two planes

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0.$$

Elliptic wedge. The surface whose equation is

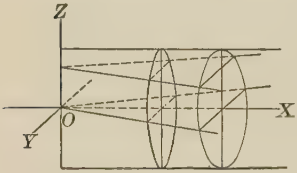

FIG. II*k*

$$x^2z^2 - b^2x^2 + a^2y^2 = 0 \qquad (47)$$

is known as an *elliptic wedge.*

The plane $x = x_0$, when x_0 is different from 0, cuts it in the ellipse

$$\frac{z^2}{b^2} + \frac{a^2y^2}{b^2x^2} = 1, \quad x = x_0,$$

whose axes are parallel to the z- and y-axes, respectively. That in the direction of the z-axis is of constant length $2b$. The length of the other is $\frac{2bx_0}{a}$.

If $x_0 > a$, the major axis of the ellipse is horizontal.

If $x_0 = a$, the section is a circle.

If $0 < x_0 < a$, the major axis of the section is vertical.

If $x_0 = 0$, the section is the segment of the z-axis from $z = -b$ to $z = b$, which may be considered as a limiting case of an ellipse.

The plane $z = z_0$ cuts the surface in the pair of lines

$$(z_0^2 - b^2)x^2 + a^2y^2 = 0, \quad z = z_0.$$

If $|z_0| < b$, the lines are real and distinct.

If $z_0 = \pm b$, the lines are coincident and lie in the xz-plane.

If $|z_0| > b$, the lines are imaginary.

The surface lies entirely between the planes $z = -b$ and $z = b$.

It is symmetrical with respect to the coördinate planes, the coördinate axes, and the origin.

In the figure, a part of half the surface is illustrated.

The elliptic wedge is a special case of a *right conoid.* The latter is a surface generated by a straight line which always intersects a given line orthogonally and satisfies some other condition. In the case of the elliptic wedge this other condition is that the generating line should always meet an ellipse which has one of its axes parallel to the given line, but its plane does not contain the given line.

It is not difficult to show that the equations

$$x = u \cos v, \quad y = u \sin v, \quad z = \phi(v), \tag{48}$$

where $\phi(v)$ is any function of v, represent a right conoid. In particular, if $z = \phi(v) = \sqrt{b^2 - a^2 \tan^2 v}$, the right conoid is the elliptic wedge (47). For from the first two equations of (48) we have $\tan v = \frac{y}{x}\cdot$ Putting this in the value of z we have

$$z^2 = b^2 - \frac{a^2y^2}{x^2} \quad \text{or} \quad x^2z^2 - b^2x^2 + a^2y^2 = 0.$$

Again, if $z = a \cot v$, we have on eliminating u and v, $z = \frac{ax}{y}$ or $yz = ax$. From (44), this is seen to be an hyperbolic paraboloid.

EXERCISES

Classify the following surfaces by noting the types of sections made by various planes with them or by other considerations.

1. $x^2 + y^2 + z^2 + 2\,ax + 2\,by + 2\,cz + d = 0.$
2. $a^2x^2 + y^2 + z^2 = 1.$
3. $x^2 - y^2 - z^2 = -1.$
4. $x^2 - y^2 - z^2 = 1.$
5. $x^2 - y^2 - z^2 = 0.$
6. $y^2 + z^2 - 2\,ax = 0.$
7. $y^2 + z^2 - 2\,az = 0.$
8. $y^2 + xy - yz = 0.$
9. $x^2 + y^2 - 2\,axz = 1.$
10. $x^2 + y^2 - 2\,axz = -1.$
11. $x^2 + y^2 - 2\,axz = 0.$
12. $yz = 2\,ax.$
13. $xy + yz + zx = 0.$
14. $xz + yz = 1.$
15. $xz + yz = 0.$
16. $x^2 - (y-1)^2 + z^2 - 4\,z = 0.$
17. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$
18. $x = \sin\sqrt{y^2 + z^2}.$

III. CURVE SKETCHING

The problem of sketching a curve, whose equation is given, is facilitated by making use of the following principles:

1. Symmetry. Two points P and Q are *symmetrical with respect to a point* O if O is the middle point of the line-segment PQ.

Two points P and Q are *symmetrical with respect to a line* if the latter bisects the line-segment PQ at right angles. Each of the points is the reflection of the other, considering the line as a reflecting mirror.

A curve is *symmetrical with respect to a point* O (or *to a line*) if, corresponding to each point P of the curve, there is another one Q upon it which is symmetrical to P with respect to the point O (or to the line).

When a curve is symmetrical with respect to a point O, the latter is said to be the *center of symmetry* of the curve, or simply its *center*. When it is symmetrical with respect to a line, the latter is said to be the *axis of symmetry* of the curve, or simply its *axis*.

It will be left as an exercise to prove that if a curve is symmetrical with respect to each of two mutually perpendicular lines, it is symmetrical with respect to their point of intersection.

Symmetry with respect to the origin, to either coördinate axis, or to the line $y = x$ is obvious upon inspection of the equation of the curve $F(x, y) = 0$.

(*a*) The points (a, b) and $(-a, -b)$ are symmetrical with respect to the origin. Hence, *if each of the terms of* $F(x, y)$ *is* *of even degree in x and y jointly* (a constant term being taken as of even degree), *or if each of the terms is of odd degree in the two variables, the curve is symmetrical with respect to the origin.*

For, if (a, b) is a point on the curve, $F(a, b) = 0$. If all the terms are of even degree,

$$F(-x, -y) \equiv F(x, y). \quad \therefore F(-a, -b) = 0,$$

and $(-a, -b)$ also is a point on the curve. If all the terms are of odd degree, $F(-x, -y) \equiv -F(x, y)$, and $(-a, -b)$ also is on the curve. Thus the curves $x^2 + y^2 = a^2$, $xy = 4$, $y = x^3$, $y = \dfrac{x}{x^2+1}$ (*i.e.* $x^2y - x + y = 0$) are all symmetrical with respect to the origin.

In general, whether the equation is algebraic or not, *the curve $F(x, y) = 0$ is symmetrical with respect to the origin whenever $F(-x, -y) \equiv \pm F(x, y)$.* The following curves $y = \sin x$, $y = \tan x$, $y = \dfrac{\cos x}{x}$, $y^2 = \cos x$ have this property.

(*b*) The points (a, b) and $(-a, b)$ are symmetrical with respect to the y-axis. Hence, *whenever $F(-x, y) \equiv F(x, y)$, the curve is symmetrical with respect to the y-axis.* Thus the curves $x^2 + y^2 = a^2$, $y = x^2$, $y^3 = x^2$, $y = \dfrac{x^2}{x^2+1}$, $y = \cos x$, $y = e^{-x^2}$ are symmetrical with respect to this axis.

(*c*) Similarly, *the curve $F(x, y) = 0$ is symmetrical with respect to the x-axis, whenever $F(x, -y) \equiv F(x, y)$.* As examples may be mentioned $x^2 + y^2 = a^2$, $y^4 = x^3$, $x = \cos y$.

(*d*) *If the equation of the curve is left unaltered when the variables are interchanged, the curve is symmetrical with respect to the line $y = x$.*

The points (a, b) and (b, a) are symmetrical with respect to this line; if $F(x, y) \equiv F(y, x)$, the point (b, a) is on the curve whenever (a, b) is. The following curves are of this type: $x^2 + y^2 = a^2$, $xy = 4$, $x^3 + y^3 - 3\,axy = 0$, $(x^2 + y^2)^2 = 2\,a^2xy$.

In this connection the following theorem should be noted. (Its proof is similar to that of the one just established, and will be left as an exercise.) *The two curves* $F(x, y) = 0$ *and* $F(y, x) = 0$ *are symmetrical with respect to the line* $y = x$. As examples may be mentioned $y = 2x$ and $x = 2y$; $y = x^3$ and $x = y^3$; $y = \sin x$ and $x = \sin y$ (or $y = \sin^{-1} x$).

2. Extent of the curve. Since the square of any real number is positive, it is often possible to tell by inspection or some other means when the curve is restricted to certain parts of the plane.

3. Outstanding points. A good idea of the appearance of a curve can often be obtained from the knowledge of certain outstanding points upon it. The most important of these are the following:

(*a*) The points where the curve crosses the axes;

(*b*) the maximum and minimum points (including those with vertical tangents);

(*c*) the flexes (including those with vertical tangents);

(*d*) the singular points; see item 6 below.

If the curve extends to infinity, find in which quadrant or quadrants it leaves the diagram, and be sure to plot a sufficient number of points so that, beyond these, the curve proceeds in a known way.

4. Direction of the curve at these points. It is very helpful to know the slope of the curve at selected points. This information is given by the value of y'.

5. Concavity. From the algebraic sign of y'' the direction of the concavity of the curve at selected points can be determined.

6. Singular points. If y is defined implicitly as a function of x by the equation of the curve, $F(x, y) = 0$, its derivative is given in the form of a fraction

$$y' = \frac{\phi(x, y)}{\psi(x, y)}.$$

If both $\phi(x, y)$ and $\psi(x, y)$ vanish at some point (a, b) on the curve, the slope at that point assumes the indeterminate form $\frac{0}{0}$, and the point is said to be a *singular point* of the curve. As its name implies, such a point is an unusual one. Its two coördinates must satisfy the three equations

$$F(x, y) = 0,\ \phi(x, y) = 0,\ \psi(x, y) = 0.$$

This is possible under certain conditions only.

It will be sufficient to give a brief discussion of the case where $F(x, y)$ is a polynomial in x and y of degree n, and the origin is the singular point. If any other point (a, b) is a singular point, the origin can be transformed to that point by the transformation

$$x = x_1 + a, \quad y = y_1 + b.$$

Let the equation of the curve be

$$F(x, y) \equiv a_0 + b_0x + b_1y + c_0x^2 + 2\,c_1xy + c_2y^2 + \cdots + k_ny^n = 0. \quad (1)$$

Then

$$y' = -\frac{b_0 + 2\,c_0x + 2\,c_1y + \cdots}{b_1 + 2\,c_1x + 2\,c_2y + \cdots}. \quad (2)$$

If the origin is on the curve, $a_0 = 0$, and the slope of the tangent at the origin is $-\frac{b_0}{b_1}$. The equation of this tangent, $b_0x + b_1y = 0$, may be obtained by placing the sum of the terms of first degree in (1) equal to zero.

If, in (1), $a_0 = b_0 = b_1 = 0$, $y' = \frac{0}{0}$ for $x = 0$, $y = 0$, and the origin is a singular point. That is, *the origin is a singular point when the equation of the curve contains no terms of lower degree than the second.* Suppose now that the equation of the curve has the form

$$F(x, y) \equiv c_0x^2 + 2\,c_1xy + c_2y^2 + \cdots + k_ny^n = 0,$$

where at least one of the terms of second degree appears. The line[1] $y = mx$ meets the curve in the points whose abscissas are given by

$$(c_0 + 2\,c_1m + c_2m^2)x^2 + (\quad)x^3 + \cdots + (\quad)x^n = 0;$$

---

[1] If $c_2 = 0$ and $c_0 \neq 0$, use $x = ny$. If $c_0 = c_2 = 0$, use $mx = ny$. In the latter case, the equation (3) takes the form $2\,c_1mn = 0$, whence $m = 0$ and $n = 0$, separately.

that is, at least two of the points of intersection of the line, for all choices of m, lie at the origin. The origin is said to be a *double point* or *node* of the curve.

In particular, as m approaches a value for which

$$c_0 + 2\,c_1 m + c_2 m^2 = 0, \tag{3}$$

the corresponding line $y = mx$ approaches a position in which it has three of its intersections with the curve at the origin. In this limiting position it is tangent to the curve.

If the quadratic equation (3) has two real and distinct roots, there are two real and distinct tangents to the curve at the double point. The latter is said to be a *crunode.* (See Fig. III b in Ex. 3 below.)

If the roots of (3) are imaginary, the tangents at the double point are imaginary, and there are no other real points in its immediate vicinity. In this case the double point is known as an *isolated* or *conjugate point* or an *acnode.* (See Fig. III c in Ex. 4 below.)

If the roots of (3) are equal, the tangents at the double point coincide. The double point is known as a *cusp* or a *stationary point;* for, if the curve is supposed to be generated by the motion of a point, at such a point the motion is brought to a stop temporarily, and when resumed, is directed oppositely. (See Figs. III a and III e in Exs. 2 and 6, respectively.)

These are the simplest types of double points. It is not practicable to give a complete discussion of this subject here.

The aggregate of the terms of second degree equated to zero,

$$c_0 x^2 + 2\,c_1 xy + c_2 y^2 = 0, \tag{4}$$

is the equation of the pair of tangents to the curve at the double point. Why? The equation of each of the tangents may be obtained by equating one or the other of the factors of the left-hand member of (4) to zero.

If in (1), the equation of the curve, there are no terms of lower degree than the third, at least three of the points in which the line $y = mx$ cuts the curve are located at the origin for all choices of m. The curve has a *triple point* there; and the aggregate of the terms of third degree equated to zero is the equation of the tangents to the curve at the triple point.

The discussion can be extended to the study of the existence of singular points of higher order of multiplicity at the origin, and of the properties of the curve at such a point.

7. Asymptotes. If the tangent to a curve approaches coincidence with a definite line as its point of contact with the curve moves to infinity, this line is known as an *asymptote* to the curve. The asymptotes of an hyperbola furnish a familiar illustration. The parabola has no asymptote, because the entire tangent to the curve goes to infinity with the point of contact. The sine curve, $y = \sin x$, has no asymptote, because the tangent to the curve has no limiting position as the point of contact goes to infinity.

The existence of *vertical* and *horizontal* asymptotes can, as a rule, be determined readily from the equation of the curve, especially when the equation can be put in the form $y = f(x)$. These arise when y approaches infinity as x approaches a finite value, and when y approaches a finite value as x approaches infinity, respectively. Thus in the case of $y = \frac{x-1}{x}$, $y = \infty$ for $x = 0$, whence $x = 0$ is a vertical asymptote; and y approaches 1 as x approaches ∞, whence $y = 1$ is a horizontal asymptote.

In the case of a vertical asymptote, y and y' both approach infinity as x approaches some finite value. In the case of a horizontal asymptote, y approaches some finite value, including zero, and y' approaches zero as x approaches infinity. In the case of an oblique asymptote, y approaches infinity and y' approaches a finite value, different from zero, as x approaches infinity. If y approaches infinity with x, and y' approaches either zero or infinity, there is no asymptote. The various possibilities may be presented schematically as follows.

The values indicated are the limiting values. A finite value is represented by a.

x	y	y'	TYPE OF ASYMPTOTE
a	∞	∞	Vertical
∞	a	0	Horizontal
∞	∞	$a \neq 0$	Oblique
∞	∞	0 or ∞	None

Thus at least one of the variables becomes infinite in the case of an asymptote. But ∞ is not a number and is not subject to the laws of algebra. Hence it is evident that a complete discussion of the subject cannot be given without the aid of additional analytic apparatus, which it is not advisable to introduce here. The following facts will, however, be found to suffice in many cases:

It can be proved that, in general, two equations in x and y of the mth and nth degree respectively have mn common solutions. Geometrically this says that two curves of the mth and nth orders respectively intersect in mn points. To find these, we may eliminate one of the variables between the two equations, obtaining thus an equation in the remaining variable, which, in general, is of the mnth degree. Having found the roots of this, the corresponding values of the other variable must then be found by using the given equations. While this is not always a practical method, theoretically it is a possible one.

If the equation in the one variable, resulting from eliminating the other, is of lower degree than mn, the number of intersections of the curves is less than mn. If its degree is $mn - k$, it is customary to say that *the curves have k intersections at infinity.* Thus two straight lines in general meet in a point. But if we take two parallel lines $y = mx + b$ and $y = mx + b'$, on eliminating y between them, we have for the resulting equation in x, $b' - b = 0$, which is of zero degree in x, instead of the first degree. To say that "two parallel lines meet at infinity" does not imply the usual type of intersection, for the distance between two parallel lines is always the same throughout their entire extent. It is because this form of language is consistent with results obtained after the introduction of a new kind of analytic apparatus that it is justified. Another possible justification of it is the following: The two lines $y = mx + b$ and $y = m'x + b'$ intersect in the point whose abscissa is $x = -\frac{b - b'}{m - m'}$. As m' approaches m as a limit, assuming that $b' - b \neq 0$, the resulting value of x approaches ∞.

Similarly the line $y = mx + b$ meets the hyperbola $x^2 - y^2 = 1$ in the points whose abscissas are the roots of

$$(1 - m^2)x^2 - 2\,bmx - b^2 - 1 = 0. \tag{5}$$

In general there are two such roots, and consequently two points of intersection. But when $m = 1$, the resulting equation in x is of the first degree. The point $\left(-\frac{b^2 + 1}{2\,b}, \frac{b^2 - 1}{2\,b}\right)$ is the only finite point of intersection. The other is said to be "at infinity." In particular, if $b = 0$, the equation in x reduces to $-1 = 0$, which is of zero degree in x. Both of the points of intersection are said to have "gone to infinity" and the line $y = x$ is tangent to the hyperbola there; that is, it is an asymptote.

In general, then, the process for finding an oblique or horizontal asymptote is the following: Let $F(x, y) = 0$ be the equation of the curve, and let it be of degree n in x and y jointly. Whenever possible, determine m and b so that the result of eliminating y between $F(x, y) = 0$ and $y = mx + b$ is an equation in x of degree $n - 2$ at most.

Thus, in the case of the hyperbola $x^2 - y^2 = 1$, we saw that equation (5) results from eliminating y between the equation of the curve and $y = mx + b$. Putting $1 - m^2 = 0$ and $2\,bm = 0$, we have

$$m = \pm 1 \quad \text{and} \quad b = 0.$$

Each of the lines $y = x$ and $y = -x$ is an asymptote.

Again, in the case of the folium of Descartes, $x^3 + y^3 = 3\,axy$ (page 71), on eliminating y between this equation and $y = mx + b$, we have

$$(m^3 + 1)x^3 + 3(bm^2 - am)x^2 + 3(b^2m - ab)x + b^3 = 0.$$

Putting $m^3 + 1 = 0$ and $bm^2 - am = 0$, we have

$$m = -1 \quad \text{and} \qquad b = -a.$$

The line $x + y + a = 0$ is an asymptote.

In the case of the parabola $y^2 = 4\,ax$, the result of eliminating y between this equation and $y = mx + b$ is

$$m^2x^2 + (2\,bm - a)x + b^2 = 0.$$

It is not possible to find values of m and b so as to satisfy

$$m^2 = 0 \quad \text{and} \quad 2\,bm - a = 0$$

simultaneously. There is no asymptote.

The method just given does not apply to vertical asymptotes, for the equation of a vertical line is of the form $x = a$. To find a vertical asymptote, we eliminate x between $F(x, y) = 0$ and $x = a$, and so determine a, when possible, that the resulting equation in y, $F(a, y) = 0$, is at most of degree $n - 2$. Evidently, if $F(x, y)$ contains a term in y^n, there can be no vertical asymptote, for the coefficient of y^n in $F(a, y)$ will not contain a, and hence cannot be made equal to zero.

The curve $xy^2 - y^2 = x^3 + x^2 + 1$ has a vertical asymptote; for putting $x = a$ in the equation of the curve,

$$(a - 1)y^2 = a^3 + a^2 + 1.$$

For $a = 1$, $x = a$ has all its intersections with the curve at infinity. The line is an asymptote.

It will be left as an exercise to show also that $y = x + 1$ and $y = -x - 1$ are asymptotes.

Remark. Every line $y = mx + b$ or $x = a$ which meets a curve in two or more points at infinity is not necessarily an asymptote, just as a line which has two or more of its intersections with a curve at a given point need not be tangent to the curve at that point. If the curve has a double point, for example, every line through that point not a tangent there will have two of its intersections with the curve located there. So a curve may have singular points at infinity. In general we may say, in order that $y = mx + b$ (or $x = a$) may be an asymptote, there must be one or more special values of b (or of a) for which the line has more points of intersection at infinity with the curve than for a general value of b (or of a).

Thus $y = x + b$ has one finite and one infinite point of intersection with $x^2 - y^2 = 1$, for all values of b except for $b = 0$. In the latter case both points of intersection are at infinity, and $y = x$ is an asymptote.

Again $x = a$ has one finite and two infinite points of intersection with $y = x^3$, for all values of a. There is no vertical asymptote for this curve. As a matter of fact, it is not difficult to show that this curve has no asymptotes of any sort.

Some of the principles enumerated above were employed in sketching the curves appearing in §§ 18, 19, 21. Additional applications will be made in the following illustrative exercises. We shall be interested only in real points of the curves.

Ex. 1. $y = ax^3 + 3\,bx^2 + 3\,cx + d.$

This curve is known as a *cubic parabola*. For large values of x, the term of highest degree, ax^3, determines the sign of y. Supposing the curve to be generated by a point moving along it from left to right, the curve enters the diagram by the third quadrant and leaves it by the first, when a is positive; when a is negative the curve enters by the second quadrant and leaves by the fourth.

From $y' = 3(ax^2 + 2\,bx + c)$ we see that

- if the roots of $ax^2 + 2\,bx + c = 0$ are real and distinct, the curve has a maximum and a minimum point; why?
- if the roots of $ax^2 + 2\,bx + c = 0$ are real and equal, or if they are imaginary, there are no real maximum and minimum points; why?

From $y'' = 6\,(ax + b)$ it follows that there is always one real flex, which has a horizontal tangent if $ax^2 + 2\,bx + c = 0$ has equal roots. On transforming the origin to the flex, it is readily seen, from the resulting form of the equation, that every cubic parabola is symmetrical with respect to its flex.

By determining the points where the curve meets the axes, the flex, the maximum and minimum points, if there are any, and several additional points, along with the slopes of the tangents at these points, the curve is readily sketched.

The three types of cubic parabola are illustrated by Figs. 11, 7, 13 (pages 52, 47, 53), respectively.

Ex. 2. $y^2 = 2x^3$.

This curve is known as a *semi-cubical parabola*. The name is suggested by the form of the equation when solved for y, thus

$$y = \pm \sqrt{2}\, x^{\frac{3}{2}}.$$

The curve may be looked upon as consisting of two branches, one for each of the two signs. The two branches are symmetrical with respect to the x-axis; one lies entirely in the first quadrant, and the other in the fourth, since y is imaginary for negative values of x. From

$$y' = \pm \frac{3x^{\frac{1}{2}}}{\sqrt{2}},$$

it is evident that the branch in the first quadrant has a positive slope at all points, except at the origin, where the slope is zero; similarly the branch in the fourth quadrant has a negative slope at all points, except at the origin, where the slope is zero also. The two branches terminating in the origin with a common tangent form a cusp there.

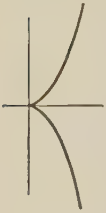

FIG. III*a*

This fact can also be obtained from the first form of the equation. Since there are no terms of degree lower than the second, we know there must be a double point at the origin. The equation of the tangents there is $y^2 = 0$. Since the two tangents coincide, we have a cusp.

To sketch the curve, we note from the form of y'' that there are no flexes. Hence the additional points $(\frac{1}{2}, \pm\frac{1}{2})$, $(1, \pm\sqrt{2})$, $(2, \pm 4)$, with the corresponding slopes $\pm\frac{3}{2}$, $\pm\dfrac{3\sqrt{2}}{2}$, ± 3 at these points, are sufficient to give a good idea of the curve. See Fig. III *a*.

Ex. 3. $y^2 = x(x-4)^2$.

Moving the origin to the point $(4, 0)$, the equation takes the form $y^2 = (x_1+4)x_1^2 = x_1^3 + 4x_1^2$. The new origin is a double point, and the equation of the tangents to the curve at this point is $y^2 = 4x_1^2$. The slopes of these lines are ± 2 respectively. The curve is symmetrical with respect to the x-axis. To sketch it, solve for y.

$$y = \pm\sqrt{x}(x-4).$$

No part of the curve lies to the left of the y-axis. There are two branches of the curve, symmetrical with respect to the x-axis. Each branch meets this axis in the points $(0, 0)$ and $(4, 0)$. Both branches have an infinite slope at $(0, 0)$, while at $(4, 0)$ their respective slopes are 2 and -2, as may be seen from

$$y' = \pm\sqrt{x} \pm \frac{x-4}{2\sqrt{x}},$$

and as was noted above.

Corresponding to $x = \frac{4}{3}$, the branch using the $+$ sign has a minimum and the other has a maximum.

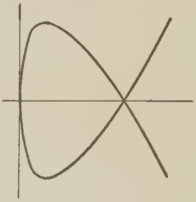

FIG. III*b*

It will be left as an exercise to show that the curve has no real flexes.

If, in addition to the points and corresponding slopes already mentioned, we note that the curve passes through the points $(6, \pm 4.9)$ with the corresponding slopes ± 2.7, the curve is readily seen to be as indicated in Fig. III *b*. It is a cubic with a crunode.

Ex. 4. $y^2 = x^2(x-4)$.

Writing the equation in the form

$$4x^2 + y^2 - x^3 = 0,$$

we see that the origin is a double point, and the equation of the tangent lines there is $4x^2 + y^2 = 0$. The tangents are imaginary. From the equation of the curve it is obvious that, excepting the origin, there are no real points of the curve to the left of the line $x = 4$. The origin is an isolated or conjugate point or acnode.

The curve is symmetrical with respect to the x-axis.

It will be left as an exercise to show that there are no real maximum and minimum points, that the points $(\frac{16}{3}, \pm \frac{32}{9}\sqrt{3})$ are flexes, with slopes $\pm 2\sqrt{3}$ respectively, and that the curve is as indicated in Fig. III *c*. It is a cubic with an acnode.

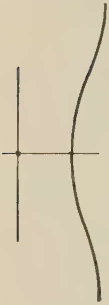

FIG. III *c*

Ex. 5. $y^2 = x(x^2 - 1)$.

The curve is symmetrical with respect to the x-axis. There are no real points on it to the left of the line $x = -1$, or between the lines $x = 0$ and $x = 1$. It will be left as an exercise to show that there is a maximum and a minimum point corresponding to $x = -\frac{1}{3}\sqrt{3}$, and two flexes corresponding to $x = 1.47$, and that the curve looks as indicated in Fig. III *d*. It is a cubic with an oval.

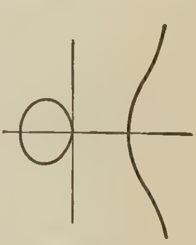

FIG. III *d*

The cubics treated in Exs. 2–5 can be grouped under the general form

$$y^2 = A(x-a)(x-b)(x-c).$$

If $a = b = c$, the curve is a semi-cubical parabola with its cusp at $(a, 0)$;

if $a < b = c$, the curve is a cubic with a crunode at $(b, 0)$;

if $a = b < c$, the curve is a cubic with an acnode at $(a, 0)$;

if $a < b < c$, the curve is a cubic with an oval extending from $(a, 0)$ to $(b, 0)$.

Ex. 6. $y^2 = \dfrac{x^3}{2a - x}$.

It will be left as an exercise to show that the curve is symmetrical with respect to the x-axis;
it has a cusp at the origin;
it lies between the lines $x = 0$ and $x = 2a$;
it has no real maximum or minimum points or flexes;
it has the line $x = 2a$ as a vertical asymptote;
a sketch of it appears in Fig. III *e*.

The curve is known as the *cissoid of Diocles.* It was employed in an attempt at solving the classic problem of the "duplication of the cube," where it is required to construct, by means of ruler and compass only, the edge of a cube whose content is twice that of a given cube. A simple definition of the curve, by means of which points on it can be constructed readily, is the following:

Starting with a circle, draw the diameter OA and the tangent at A. See Fig. III *e*. Any other line through O meets the circle in a second point, say Q, and the tangent in some point, T. If the length OP, equal to QT, is taken as a radius vector along the line from O as an origin, P is a point on the cissoid.

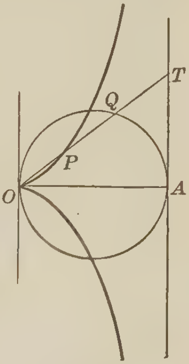

FIG. III *e*

As an exercise, find the equation of the curve from this definition.

Ex. 7. $y = \dfrac{8a^3}{4a^2 + x^2}$.

It will be left as an exercise to show that
the curve is symmetrical with respect to the y-axis;
it lies entirely above the x-axis and has the latter as a horizontal asymptote;
the point $(0, 2a)$ is a maximum;

the points $\left(\pm\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$ are flexes;

a sketch of the curve appears in Fig. III *f*.

The curve is known as the *witch of Agnesi*. It has been known for a long time. But its name is associated with that of the Italian mathematician Maria Gaetena Agnesi (1718–1799), who suggested the following method of constructing points on it:

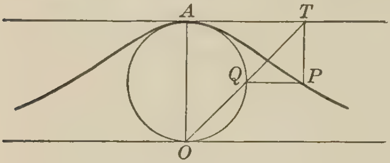

Fig. III *f*

Starting with a circle, draw the diameter OA and the tangent at A. See Fig. III *f*. Any other line through O meets the circle in a second point, say Q, and the tangent in some point, T. The intersection of the horizontal line QP with the vertical line TP is a point on the curve. As an exercise, find the equation of the curve from this definition.

Ex. 8. $(x+y)^2 = x^3$.

The origin is a cusp, and the tangent at the cusp has the equation $x + y = 0$.

The curve is not symmetrical with respect to the origin or either axis.

There are no real points for negative values of x. Solving for y,

$$y = \pm x^{\frac{3}{2}} - x.$$

$$\therefore y' = \pm \tfrac{3}{2} x^{\frac{1}{2}} - 1, \text{ and } y'' = \pm \tfrac{3}{4} x^{-\frac{1}{2}}.$$

The branch of the curve obtained by using the + sign has a minimum at $(\frac{4}{9}, -\frac{4}{27})$. It has no flexes. The other branch has no maximum or minimum points or flexes. Neither branch has an asymptote.

A method for sketching the curve rapidly is the following, which can be employed whenever the ordinate of the given curve is known as the algebraic sum of the ordinates of two easily sketched curves.

Sketch the semi-cubical parabola $y = \pm x^{\frac{3}{2}}$ and the line $y = -x$. By taking the algebraic sum of the ordinate of each branch of the semi-cubical parabola and that of the line for a selected value of x, the ordinate of a branch of the required curve for that value of x is obtained. The curve in Fig. III g was constructed in this way. While the sketch may not be as accurate as when the curve is drawn directly, it gives a good idea of how the curve looks.

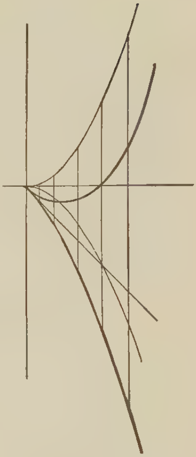

FIG. III g

EXERCISES

Discuss and sketch the following curves:

1. $y = ax^n$, where n is an integer.

Consider separately the cases when (a) n is even or odd; (b) n is positive or negative; (c) a is positive or negative. Assign numerical values and sketch.

2. $y = x^2 - 4x + 5$.

3. $y^2 = x^2 - 4x + 5$.

4. $y = 3x^2 - x^3$.

5. $y^2 = 3x^2 - x^3$.

6. $y = 3x^3 - 6x^2 + 2x$.

7. $y^2 = 3x^3 - 6x^2 + 2x$.

8. $3y = x^3 + 9x$.

9. $9y^2 = x^3 + 9x$.

10. $9y = x^4 - 4x^3$.

11. $9y^2 = x^4 - 4x^3$.

12. $9y^2 = 4x^3 - x^4$.

13. $y^4 = 4x^3 - x^4$.

14. $10y = x^5 - 10x^3 + 40$.

15. $4y^2 = (x^2 + 1)^2(4 - x^2)^3$.

16. $y^3 = 3x^2 - x^3$.

17. $y = \dfrac{x}{4 - x^2}$.

18. $y^2 = \dfrac{x}{4 - x^2}$.

19. $y = \dfrac{x^2}{4 - x^2}$.

20. $y^2 = \dfrac{x^2}{4 - x^2}$.

21. $y = \dfrac{x^3}{4 - x^2}$.

22. $y^2 = \dfrac{x^3}{4 - x^2}$.

23. $x^4 - x^2y + y^3 = 0.$

24. $x^2y + xy^2 = a^3.$

25. $x^2y^2 = a^2(x^2 + y^2).$

26. $y = \log x.$

27. $y = e^x.$

28. $y = e^{-x}.$

29. $y = e^{-x^2}.$

30. $y = e^{-x} \cos x.$

31. $y = \sin x + \cos x.$

32. $y = 2 \sin x - \cos x.$

33. $y = 2 \sin x + \cos 2x.$

34. $y = \sec x - \tan x.$

35. $y = \frac{\sin x}{x}.$

IV. TABLE OF INTEGRALS

I. ELEMENTARY PRINCIPLES

1. $$\int a\,f(x)\,dx = a\int f(x)\,dx.$$

2. $$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots]\,dx.$$
$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \cdots.$$

3. $$\int u\,dv = uv - \int v\,du.$$

II. RATIONAL ALGEBRAIC FUNCTIONS

A. Involving $a + bx$

4. $$\int x^m\,dx = \frac{x^{m+1}}{m+1}, \text{ if } m \neq -1.$$

5. $$\int \frac{dx}{x} = \log x.$$

6. $$\int (a+bx)^m\,dx = \frac{(a+bx)^{m+1}}{(m+1)b}, \text{ if } m \neq -1.$$

7. $$\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx).$$

8. $$\int x^n(a+bx)^m\,dx = \frac{1}{b^{n+1}}\int u^m(u-a)^n\,du, \text{ where } u = a+bx.$$

9. $$\int \frac{x^n\,dx}{(a+bx)^m} = \frac{1}{b^{n+1}}\int \frac{(u-a)^n}{u^m}\,dx, \text{ where } u = a+bx.$$

Particular cases of this are 10–16.

10. $\displaystyle\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$

11. $\displaystyle\int \frac{dx}{(a+bx)^3} = -\frac{1}{2\,b\,(a+bx)^2}.$

12. $\displaystyle\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}[(a+bx) - a\log(a+bx)].$

13. $\displaystyle\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right].$

14. $\displaystyle\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$

15. $\displaystyle\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}[\tfrac{1}{2}(a+bx)^2 - 2\,a(a+bx) + a^2\log(a+bx)].$

16. $\displaystyle\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[(a+bx) - 2\,a\log(a+bx) - \frac{a^2}{a+bx}\right].$

17. $\displaystyle\int \frac{dx}{x^n(a+bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(v-b)^{m+n-2}}{v^m}\,dv,$ where
$a + bx = xv.$

Particular cases of this are 18–21.

18. $\displaystyle\int \frac{dx}{x(a+bx)} = \frac{1}{a}\log\frac{x}{a+bx}.$

19. $\displaystyle\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2}\log\frac{x}{a+bx}.$

20. $\displaystyle\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$

21. $\displaystyle\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2\,bx}{a^2x(a+bx)} + \frac{2\,b}{a^3}\log\frac{a+bx}{x}.$

B. *Involving* $a^2 + x^2$, $a^2 - x^2$, $b + cx^n$.

22. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

23. $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2\,a}\log\frac{a+x}{a-x}$ or $\displaystyle\frac{1}{2\,a}\log\frac{x+a}{x-a}$ or $\displaystyle\frac{1}{a}\tanh^{-1}\frac{x}{a}$ or $\displaystyle\frac{1}{a}\coth^{-1}\frac{x}{a}.$

24. $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2\,a}\log\frac{x-a}{x+a}$ or $\displaystyle\frac{1}{2\,a}\log\frac{a-x}{a+x}$ or $\displaystyle-\frac{1}{a}\tanh^{-1}\frac{x}{a}$ or $\displaystyle-\frac{1}{a}\coth^{-1}\frac{x}{a}.$

25. $\displaystyle\int\frac{dx}{b+cx^2}=\frac{1}{\sqrt{bc}}\tan^{-1}x\sqrt{\frac{c}{b}}$, if $b>0$, $c>0$.

26. $\displaystyle\int\frac{dx}{b+cx^2}=\frac{1}{2\sqrt{-bc}}\log\frac{\sqrt{b}+x\sqrt{-c}}{\sqrt{b}-x\sqrt{-c}}$ or $\displaystyle\frac{1}{\sqrt{-bc}}\tanh^{-1}x\sqrt{\frac{-c}{b}}$, if $b>0$, $c<0$.

27. $\displaystyle\int\frac{dx}{(b+cx^2)^{m+1}}=\frac{x}{2\,mb\,(b+cx^2)^m}+\frac{2\,m-1}{2\,mb}\int\frac{dx}{(b+cx^2)^m}.$

28. $\displaystyle\int\frac{x\,dx}{b+cx^2}=\frac{1}{2\,c}\log\,(b+cx^2).$

29. $\displaystyle\int\frac{x\,dx}{(b+cx^2)^{m+1}}=\frac{-1}{2\,cm\,(b+cx^2)^m}.$

30. $\displaystyle\int\frac{x^2\,dx}{b+cx^2}=\frac{x}{c}-\frac{b}{c}\int\frac{dx}{b+cx^2}.$

31. $\displaystyle\int\frac{x^2\,dx}{(b+cx^2)^{m+1}}=\frac{-x}{2\,mc\,(b+cx^2)^m}+\frac{1}{2\,mc}\int\frac{dx}{(b+cx^2)^m}.$

32. $\displaystyle\int\frac{dx}{x\,(b+cx^2)}=\frac{1}{2\,b}\log\frac{x^2}{b+cx^2}.$

33. $\displaystyle\int\frac{dx}{x^2\,(b+cx^2)}=-\frac{1}{bx}-\frac{c}{b}\int\frac{dx}{b+cx^2}.$

34. $\displaystyle\int\frac{dx}{x^2(b+cx^2)^{m+1}}=\frac{1}{b}\int\frac{dx}{x^2(b+cx^2)^m}-\frac{c}{b}\int\frac{dx}{(b+cx^2)^{m+1}}.$

35. $$\int\frac{dx}{b+cx^3}=\frac{k}{3b}\left[\log\frac{x+k}{\sqrt{x^2-kx+k^2}}+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right],$$
where $k=\left(\frac{b}{c}\right)^{\frac{1}{3}}$.

36. $$\int\frac{x\,dx}{b+cx^3}=\frac{1}{3ck}\left[\log\frac{\sqrt{x^2-kx+k^2}}{x+k}+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right],$$
where $k=\left(\frac{b}{c}\right)^{\frac{1}{3}}$.

37. $$\int\frac{dx}{(b+cx^n)^{p+1}}=\frac{1}{b}\int\frac{dx}{(b+cx^n)^p}-\frac{c}{b}\int\frac{x^n\,dx}{(b+cx^n)^{p+1}}.$$

38. $$\int\frac{x^m\,dx}{(b+cx^n)^{p+1}}$$
$$=\frac{x^{m-n+1}}{c(m-n-np+1)(b+cx^n)^p}-\frac{b(m-n+1)}{c(m-n-np+1)}\int\frac{x^{m-n}\,dx}{(b+cx^n)^{p+1}}$$
$$=\frac{x^{m+1}}{bnp(b+cx^n)^p}-\frac{m-np+1}{bnp}\int\frac{x^m\,dx}{(b+cx^n)^p}.$$

39. $$\int\frac{dx}{x(b+cx^n)}=\frac{1}{bn}\log\frac{x^n}{b+cx^n}.$$

40. $$\int\frac{dx}{x^m(b+cx^n)^{p+1}}$$
$$=-\frac{1}{(m-1)bx^{m-1}(b+cx^n)^p}-\frac{(m+np-1)c}{(m-1)b}\int\frac{dx}{x^{m-n}(b+cx^n)^{p+1}}$$
$$=\frac{1}{bnpx^{m-1}(b+cx^n)^p}+\frac{m+np-1}{bnp}\int\frac{dx}{x^m(b+cx^n)^p}.$$

41. $$\int x^m(b+cx^n)^{p+1}dx$$
$$=\frac{x^{m-n+1}(b+cx^n)^{p+2}}{c(np+m+n+1)}-\frac{b(m-n+1)}{c(np+m+n+1)}\int x^{m-n}(b+cx^n)^{p+1}dx$$
$$=\frac{x^{m+1}(b+cx^n)^{p+1}}{np+m+n+1}+\frac{bn(p+1)}{np+m+n+1}\int x^m(b+cx^n)^p dx.$$

C. Involving $X \equiv \alpha + \beta x + \gamma x^2$.

Completing the square by adding $\frac{\beta^2}{4\gamma}$, and then subtracting this quantity, X may be written

$$\alpha - \frac{\beta^2}{4\gamma} + \gamma\left(x + \frac{\beta}{2\gamma}\right)^2.$$

Letting $b = \alpha - \frac{\beta^2}{4\gamma}$, $c = \gamma$, $y = x + \frac{\beta}{2\gamma}$, X may be replaced by

$$b + cy^2.$$

Formulas of B can then be used.

D. Rational fractions.

A rational fraction, different from any of those given above, should be expressed as a sum of simpler ones, by the methods of § 53.

III. IRRATIONAL ALGEBRAIC FUNCTIONS

A. Involving $\sqrt{a + bx}$.

42. $$\int \sqrt{a + bx}\, dx = \frac{2}{3b}\sqrt{(a + bx)^3}.$$

43. $$\int x\sqrt{a + bx}\, dx = -\frac{2(2a - 3bx)\sqrt{(a + bx)^3}}{15b^2}.$$

44. $$\int x^2\sqrt{a + bx}\, dx = \frac{2(8a^2 - 12abx + 15b^2x^2)\sqrt{(a + bx)^3}}{105b^3}.$$

45. $$\int x^m\sqrt{a + bx}\, dx$$
$$= \frac{2}{b(2m + 3)}\left[x^m\sqrt{(a + bx)^3} - ma\int x^{m-1}\sqrt{a + bx}\, dx.\right.$$

46. $$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}.$$

47. $$\int \frac{dx}{x\sqrt{a + bx}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}, \text{ if } a > 0.$$

48. $$\int\frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}},\ \text{if } a<0.$$

49. $$\int\frac{dx}{x^2\sqrt{a+bx}}=-\frac{\sqrt{a+bx}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$$

50. $$\int\frac{dx}{x^{n+1}\sqrt{a+bx}}$$
$$=-\frac{\sqrt{a+bx}}{nax^n}-\frac{(2n-1)b}{2na}\int\frac{dx}{x^n\sqrt{a+bx}},\quad n>0.$$

51. $$\int\frac{x\,dx}{\sqrt{a+bx}}=-\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

52. $$\int\frac{x^2\,dx}{\sqrt{a+bx}}=\frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

53. $$\int\frac{x^m\,dx}{\sqrt{a+bx}}$$
$$=\frac{2x^m\sqrt{a+bx}}{(2m+1)b}-\frac{2ma}{(2m+1)b}\int\frac{x^{m-1}}{\sqrt{a+bx}}dx,\quad m\neq-\tfrac{1}{2}.$$

54. $$\int\frac{\sqrt{a+bx}}{x}dx=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}.$$

55. $$\int\frac{\sqrt{a+bx}}{x^{n+1}}dx$$
$$=-\frac{1}{na}\left[\frac{\sqrt{(a+bx)^3}}{x^n}+\frac{(2n-3)b}{2}\int\frac{\sqrt{a+bx}}{x^n}dx,\right.\quad n>0.$$

56. $$\int f(x,\sqrt[n]{a+bx})\,dx$$
$$=\frac{n}{b}\int f\left(\frac{y^n-a}{b},\,y\right)y^{n-1}\,dy,\ \text{where } a+bx=y^n.$$

B. Involving $\sqrt{x^2\pm a^2}$, $\sqrt{a^2-x^2}$.

These are special forms of the more general integral given in section *C*, page 549.

In the forms indicated by an *, the logarithm may be replaced by the corresponding inverse hyperbolic function. The corresponding forms are:

$\log(x+\sqrt{x^2+a^2})$ and $\sinh^{-1}\frac{x}{a}$; $\log(x+\sqrt{x^2-a^2})$ and $\cosh^{-1}\frac{x}{a}$;

$\log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right)$ and $\sinh^{-1}\frac{a}{x}$; $\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$ and $\cosh^{-1}\frac{a}{x}$.

57. $\int\sqrt{x^2\pm a^2}\,dx=\frac{1}{2}\,x\sqrt{x^2\pm a^2}\pm\frac{1}{2}\,a^2\log\left(x+\sqrt{x^2\pm a^2}\right).$*

58. $\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\,x\sqrt{a^2-x^2}+\frac{1}{2}\,a^2\sin^{-1}\frac{x}{a}.$

59. $\int\frac{dx}{\sqrt{x^2\pm a^2}}=\log\left(x+\sqrt{x^2\pm a^2}\right).$*

60. $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}.$

61. $\int x\sqrt{x^2\pm a^2}\,dx=\frac{1}{3}\sqrt{(x^2\pm a^2)^3}.$

62. $\int x\sqrt{a^2-x^2}\,dx=-\frac{1}{3}\sqrt{(a^2-x^2)^3}.$

63. $\int\frac{x\,dx}{\sqrt{x^2\pm a^2}}=\sqrt{x^2\pm a^2}.$ 64. $\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}.$

65. $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}=\frac{1}{a}\cos^{-1}\frac{a}{x}.$

66. $\int\frac{dx}{x\sqrt{a^2\pm x^2}}=-\frac{1}{a}\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right).$*

67. $\int\frac{\sqrt{x^2-a^2}}{x}\,dx=\sqrt{x^2-a^2}-a\cos^{-1}\frac{a}{x}.$

68. $\int\frac{\sqrt{a^2\pm x^2}}{x}\,dx=\sqrt{a^2\pm x^2}-a\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right).$*

69. $\int x^2\sqrt{x^2\pm a^2}\,dx$

$$=\frac{x}{4}\sqrt{(x^2\pm a^2)^3}\mp\frac{a^2}{8}x\sqrt{x^2\pm a^2}-\frac{a^4}{8}\log\left(x+\sqrt{x^2\pm a^2}\right).*$$

70. $\int x^2\sqrt{a^2-x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2-x^2)^3}+\frac{a^2}{8}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right).$$

71. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2x}.$

72. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$

73. $\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{2}\,x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}\,a^2\log\left(x+\sqrt{x^2 \pm a^2}\right).$*

74. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}\,x\sqrt{a^2-x^2}+\frac{1}{2}\,a^2\sin^{-1}\frac{x}{a}.$

75. $\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\,dx = -\frac{\sqrt{x^2 \pm a^2}}{x}+\log\left(x+\sqrt{x^2 \pm a^2}\right).$*

76. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}.$

77. $\int \sqrt{(x^2 \pm a^2)^3}\,dx$

$$= \frac{1}{4}\,x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3}{8}\,a^2x\sqrt{x^2 \pm a^2}+\frac{3}{8}\,a^4\log\left(x+\sqrt{x^2 \pm a^2}\right).^*$$

78. $\int \sqrt{(a^2-x^2)^3}\,dx$

$$= \frac{1}{4}\,x\sqrt{(a^2-x^2)^3}+\frac{3}{8}\,a^2x\sqrt{a^2-x^2}+\frac{3}{8}\,a^4\sin^{-1}\frac{x}{a}.$$

79. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}}dx = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

80. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

81. $\int \frac{x^2dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}}+\log\left(x+\sqrt{x^2 \pm a^2}\right).$*

82. $\int \frac{x^2dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\frac{x}{a}.$

83. $$\int\sqrt{(x^2\pm a^2)^n}\,dx = \frac{x\sqrt{(x^2\pm a^2)^n}}{n+1} \pm \frac{na^2}{n+1}\int\sqrt{(x^2\pm a^2)^{n-2}}\,dx.$$

84. $$\int\sqrt{(a^2-x^2)^n}\,dx = \frac{x\sqrt{(a^2-x^2)^n}}{n+1} + \frac{na^2}{n+1}\int\sqrt{(a^2-x^2)^{n-2}}\,dx.$$

85. $$\int x\sqrt{(x^2\pm a^2)^n}\,dx = \frac{\sqrt{(x^2\pm a^2)^{n+2}}}{n+2}.$$

86. $$\int x\sqrt{(a^2-x^2)^n}\,dx = -\frac{\sqrt{(a^2-x^2)^{n+2}}}{n+2}.$$

C. Involving $\sqrt{a+bx+cx^2}$.

Let $X\equiv a+bx+cx^2,\quad D\equiv 4\,ac-b^2,\quad k\equiv\dfrac{4\,c}{D},$

$l\equiv an^2-bmn+cm^2.$

87. $$\int\frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\log\left(\sqrt{X}+\frac{2\,cx+b}{2\sqrt{c}}\right),\ \text{if } c>0.$$

88. $$\int\frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2\,cx-b}{\sqrt{-D}}\right),\ \text{if } c<0.$$

89. $$\int\frac{dx}{X\sqrt{X}} = \frac{4\,cx+2\,b}{D\sqrt{X}}.$$

90. $$\int\frac{dx}{X^n\sqrt{X}} = \frac{(4\,cx+2\,b)\sqrt{X}}{(2\,n-1)DX^n} + \frac{2\,k(n-1)}{2\,n-1}\int\frac{dx}{X^{n-1}\sqrt{X}}.$$

91. $$\int\sqrt{X}\,dx = \frac{(2\,cx+b)\sqrt{X}}{4\,c} + \frac{1}{2\,k}\int\frac{dx}{\sqrt{X}}.$$

92. $$\int X^n\sqrt{X}\,dx = \frac{(2\,cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2(n+1)k}{2\,n+1}\int\frac{X^n\,dx}{\sqrt{X}}.$$

93. $$\int\frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2\,c}\int\frac{dx}{X}.$$

94. $$\int\frac{x\,dx}{X^n\sqrt{X}}=-\frac{\sqrt{X}}{(2n-1)cX^n}-\frac{b}{2c}\int\frac{dx}{X^n\sqrt{X}}.$$

95. $$\int x\sqrt{X}\,dx=\frac{X\sqrt{X}}{3c}-\frac{b}{2c}\int\sqrt{X}\,dx.$$

96. $$\int x\,X^n\sqrt{X}\,dx=\frac{X^{n+1}\sqrt{X}}{(2n+3)c}-\frac{b}{2c}\int X^n\sqrt{X}\,dx.$$

97. $$\int\frac{dx}{x\sqrt{X}}=-\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x}+\frac{b}{2\sqrt{a}}\right),\text{ if } a>0.$$

98. $$\int\frac{dx}{x\sqrt{X}}=\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx+2a}{x\sqrt{-D}}\right)\text{ if } a<0.$$

99. $$\int\frac{dx}{x\sqrt{X}}=-\frac{2\sqrt{X}}{bx},\text{ if } a=0.$$

100. $$\int\frac{dx}{x\,X^n\sqrt{X}}$$
$$=\frac{\sqrt{X}}{(2n-1)aX^n}+\frac{1}{a}\int\frac{dx}{x\,X^{n-1}\sqrt{X}}-\frac{b}{2a}\int\frac{dx}{X^n\sqrt{X}}.$$

101. $$\int\frac{\sqrt{X}\,dx}{x}=\sqrt{X}+\frac{b}{2}\int\frac{dx}{\sqrt{X}}+a\int\frac{dx}{x\sqrt{X}}.$$

102. $$\int\frac{X^n\sqrt{X}\,dx}{x}$$
$$=\frac{X^n\sqrt{X}}{2n+1}+a\int\frac{X^{n-1}\sqrt{X}\,dx}{x}+\frac{b}{2}\int X^{n-1}\sqrt{X}\,dx.$$

103. $$\int\frac{m+nx}{\sqrt{X}}\,dx=\frac{n\sqrt{X}}{c}+\frac{2cm-bn}{2c}\int\frac{dx}{\sqrt{X}}.$$

104. $$\int\frac{dx}{(m+nx)\sqrt{X}}$$
$$=-\frac{1}{\sqrt{l}}\log\left(\frac{n\sqrt{X}+\sqrt{l}}{m+nx}+\frac{bn-2cm}{2\sqrt{l}}\right),\text{ if } l>0,$$

105. $$\int \frac{dx}{(m+nx)\sqrt{X}} = \frac{1}{\sqrt{-l}} \sin^{-1}\left[\frac{(bn-2\,cm)(m+nx)+2\,l}{n(m+nx)\sqrt{-D}}\right], \text{ if } l<0.$$

106. $$\int \frac{dx}{(m+nx)\sqrt{X}} = -\frac{2\,n\sqrt{X}}{(bn-2\,cm)(m+nx)}, \text{ if } l=0.$$

107. $$\int (m+nx)\sqrt{X}\,dx = \frac{n\,X\sqrt{X}}{3\,c} - \frac{bn-2\,cm}{2\,c}\int \sqrt{X}\,dx.$$

108. $$\int \frac{\sqrt{X}\,dx}{m+nx} = \frac{\sqrt{X}}{n} + \frac{bn-2\,cm}{2\,n^2}\int \frac{dx}{\sqrt{X}} + \frac{l}{n^2}\int \frac{dx}{(m+nx)\sqrt{X}}.$$

109. $$\int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2\,n+m)\,c} - \frac{(2\,n+2\,m-1)\,b}{2\,c(2\,n+m)}\int \frac{x^{m-1}X^n dx}{\sqrt{X}} - \frac{(m-1)\,a}{(2\,n+m)\,c}\int \frac{x^{m-2}X^n dx}{\sqrt{X}}.$$

110. $$\int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}\,dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}\,dx}{X^n\sqrt{X}}.$$

111. $$\int \frac{dx}{x^m X^n\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n} - \frac{(2\,n+2\,m-3)\,b}{2\,a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}} - \frac{(2\,n+m-2)\,c}{(m-1)\,a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

112. $$\int \frac{X^n dx}{x^m\sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)x^{m-1}} + \frac{(2\,n-1)\,b}{2(m-1)}\int \frac{X^{n-1}dx}{x^{m-1}\sqrt{x}} + \frac{(2\,n-1)\,c}{m-1}\int \frac{X^{n-1}dx}{x^{m-2}\sqrt{X}}.$$

D. Miscellaneous Irrational Expressions.

113. $$\int \sqrt{2\,ax - x^2}\,dx = \tfrac{1}{2}(x-a)\sqrt{2\,ax - x^2} + \tfrac{1}{2}\,a^2 \sin^{-1}\frac{x-a}{a}.$$

114. $$\int \frac{dx}{\sqrt{2\,ax - x^2}} = \text{vers}^{-1}\frac{x}{a} = \cos^{-1}\frac{a-x}{a} = 2\sin^{-1}\sqrt{\frac{x}{2\,a}}.$$

115. $$\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)} + (a-b)\log\left(\sqrt{a+x} + \sqrt{b+x}\right).$$

116. $$\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\sin^{-1}\sqrt{\frac{x+b}{a+b}}.$$

117. $$\int \sqrt{\frac{a+x}{b-x}}\,dx = -\sqrt{(a+x)(b-x)} - (a+b)\sin^{-1}\sqrt{\frac{b-x}{a+b}}.$$

118. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = -\sqrt{1-x^2} + \sin^{-1}\sqrt{\frac{x}{a+b}}.$$

119. $$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}.$$

IV. TRANSCENDENTAL FUNCTIONS

In this section, n is a positive integer, p and q are positive or negative integers, and r and s are any numbers.

120. $$\int \sin x\,dx = -\cos x.$$

121. $$\int \sin^2 x\,dx = \tfrac{1}{2}(x - \sin x \cos x).$$

122. $$\int \sin^{2n+1} x\,dx = \int (1-\cos^2 x)^n \sin x\,dx.$$

Expand $(1-\cos^2 x)^n$ and use formula 135.

123. $$\int \sin^{2n} x\,dx = -\frac{\sin^{2n-1} x \cos x}{2\,n} + \frac{2\,n-1}{2\,n}\int \sin^{2n-2} x\,dx.$$

124. $\int \frac{dx}{\sin x} = \int \csc x \, dx.$ Use formula 172.

125. $\int \frac{dx}{\sin^{2n+1} x} = \frac{-\cos x}{2n \sin^{2n} x} + \frac{2n-1}{2n} \int \frac{dx}{\sin^{2n-1} x}.$

126. $\int \frac{dx}{\sin^{2n} x} = \int \csc^{2n} x \, dx.$ Use formula 175.

127. $\int \cos x \, dx = \sin x.$ **128.** $\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x).$

129. $\int \cos^{2n+1} x \, dx = \int (1 - \sin^2 x)^n \cos x \, dx.$ Expand, and use formula 134.

130. $\int \cos^{2n} x \, dx = \frac{\cos^{2n-1} x \sin x}{2n} + \frac{2n-1}{2n} \int \cos^{2n-2} x \, dx.$

131. $\int \frac{dx}{\cos x} = \int \sec x \, dx.$ Use formula 168.

132. $\int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n \cos^{2n} x} + \frac{2n-1}{2n} \int \frac{dx}{\cos^{2n-1} x}.$

133. $\int \frac{dx}{\cos^{2n} x} = \int \sec^{2n} x \, dx.$ Use formula 171.

134. $\int \sin^r x \cos x \, dx = \frac{\sin^{r+1} x}{r+1}, \quad r \neq -1.$

135. $\int \sin x \cos^r x \, dx = -\frac{\cos^{r+1} x}{r+1}, \quad r \neq -1.$

136. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{\sin 4x}{32}.$

137. $\int \frac{dx}{\sin x \cos x} = \log \tan x.$

138. $\int \frac{dx}{\sin^2 x \cos x} = \log(\sec x + \tan x) - \csc x.$

139. $\int \frac{dx}{\sin x \cos^2 x} = \log(\csc x - \cot x) + \sec x.$

The following four reduction formulas hold for p and q any *positive* or *negative integers*, excepting as noted:

140. $$\int \sin^p x \cos^q x \, dx$$
$$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x \, dx, \text{ if } p + q \neq 0.$$

141. $$\int \sin^p x \cos^q x \, dx$$
$$= - \frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx, \text{ if } p + q \neq 0.$$

142. $$\int \sin^p x \cos^q x \, dx$$
$$= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x \, dx, \text{ if } p \neq -1.$$

143. $$\int \sin^p x \cos^q x \, dx$$
$$= - \frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x \, dx, \text{ if } q \neq -1.$$

In the following three formulas r and s are any numbers, provided $r + s \neq 0$ and $r - s \neq 0$.

144. $$\int \sin rx \sin sx \, dx = - \frac{\sin (r+s)x}{2(r+s)} + \frac{\sin (r-s)x}{2(r-s)}.$$

145. $$\int \sin rx \cos sx \, dx = - \frac{\cos (r+s)x}{2(r+s)} - \frac{\cos (r-s)x}{2(r-s)}.$$

146. $$\int \cos rx \cos sx \, dx = \frac{\sin (r+s)x}{2(r+s)} + \frac{\sin (r-s)x}{2(r-s)}.$$

147. $$\int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}}$$
$$\text{or } \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1} \frac{b + a \sin x}{a + b \sin x}, \text{ if } a^2 > b^2.$$

148. $\displaystyle\int\frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}$

or $\displaystyle\frac{1}{\sqrt{b^2-a^2}}\log\frac{a+b\sin x}{b+a\sin x+\sqrt{b^2-a^2}\cos x},$
if $a^2<b^2$

149. $\displaystyle\int\frac{dx}{1+\sin x}=-\tan\frac{1}{2}\left(\frac{\pi}{2}-x\right)=\tan x-\sec x.$

150. $\displaystyle\int\frac{dx}{1-\sin x}=\cot\frac{1}{2}\left(\frac{\pi}{2}-x\right)=\tan x+\sec x.$

151. $\displaystyle\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$

or $\displaystyle\frac{-1}{\sqrt{a^2-b^2}}\sin^{-1}\frac{b+a\cos x}{a+b\cos x}$, if $a^2>b^2$.

152. $\displaystyle\int\frac{dx}{a+b\cos x}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}$

or $\displaystyle\frac{1}{\sqrt{b^2-a^2}}\log\frac{b+a\cos x+\sqrt{b^2-a^2}\sin x}{a+b\cos x}$

or $\displaystyle\frac{1}{\sqrt{b^2-a^2}}\tanh^{-1}\left(\frac{\sqrt{b^2-a^2}\sin x}{b+a\cos x}\right)$, if $b^2>a^2$.

153. $\displaystyle\int\frac{dx}{1+\cos x}=\tan\frac{x}{2}=\csc x-\cot x.$

154. $\displaystyle\int\frac{dx}{1-\cos x}=-\cot\frac{x}{2}=-\csc x-\cot x.$

155. $\displaystyle\int\frac{dx}{\sin x+\cos x}=\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right).$

156. $\displaystyle\int\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right).$

157. $\int \sqrt{1 - \cos x}\, dx = -2\sqrt{2} \cos \frac{x}{2}$.

158. $\int \sqrt{1 + \cos x}\, dx = 2\sqrt{2} \sin \frac{x}{2}$.

159. $\int \tan x\, dx = -\log \cos x = \log \sec x$.

160. $\int \tan^2 x\, dx = \int (\sec^2 x - 1) dx = \tan x - x$.

161. $\int \tan^{2n+1} x\, dx = \int (\sec^2 x - 1)^n \tan x\, dx$.

Expand, and use formula 176.

162. $\int \tan^{2n} x\, dx$

$$= \frac{\tan^{2n-1} x}{2n-1} - \frac{\tan^{2n-3} x}{2n-3} + \frac{\tan^{2n-5} x}{2n-5} - \cdots + (-1)^{n+1} (\tan x - x).$$

163. $\int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2} [b \log (a \cos x + b \sin x) + ax]$.

164. $\int \cot x\, dx = \log \sin x$.

165. $\int \cot^2 x\, dx = \int (\csc^2 x - 1) dx = -\cot x - x$.

166. $\int \cot^{2n+1} x\, dx = \int (\csc^2 y - 1)^n \cot x\, dx$.

Expand, and use formula 177.

167. $\int \cot x^{2n}\, dx$

$$= -\frac{\cot^{2n-1} x}{2n-1} + \frac{\cot^{2n-3} x}{2n-3} - \cdots + (-1)^n (\cot x + x).$$

168. $\int \sec x\, dx = \log \tan \frac{1}{2}\left(\frac{\pi}{2} + x\right) = \log (\sec x + \tan x)$.

169. $\int \sec^2 x\, dx = \tan x$.

170. $\int \sec^{2n+1} x\,dx = \int \frac{dx}{\cos^{2n+1} x}$. Use formula 132.

171. $\int \sec^{2n} x\,dx = \int (\tan^2 x + 1)^{n-1} \sec^2 x\,dx.$
Expand, and use formula 178.

172. $\int \csc x\,dx = \log \tan \frac{x}{2} = \log(\csc x - \cot x).$

173. $\int \csc^2 x\,dx = -\cot x.$

174. $\int \csc^{2n+1} x\,dx = \int \frac{dx}{\sin^{2n+1} x}$. Use formula 125.

175. $\int \csc^{2n} x\,dx = \int (\cot^2 x + 1)^{n-1} \csc^2 x\,dx.$
Expand, and use formula 179.

176. $\int \sec^r x \tan x\,dx = \frac{\sec^r x}{r}$, if $r \neq 0$.

177. $\int \csc^r x \cot x\,dx = -\frac{\csc^r x}{r}$, if $r \neq 0$.

178. $\int \tan^s x \sec^2 x\,dx = \frac{\tan^{s+1} x}{s+1}$, if $s \neq -1$.

179. $\int \cot^s x \csc^2 x\,dx = -\frac{\cot^{s+1} x}{s+1}$, if $s \neq -1$.

180. $\int e^{ax}\,dx = \frac{e^{ax}}{a}$.

181. $\int a^{bx}\,dx = \frac{a^{bx}}{b \log a}$.

182. $\int \log x\,dx = x \log x - x.$

183. $\int \log_a x\,dx = x \log_a x - x \log_a e.$

184. $\int x \sin ax\,dx = \frac{1}{a^2} \sin ax - \frac{1}{a} x \cos ax.$

185. $\int x^n \sin ax\,dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\,dx.$

186. $\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{1}{a} x \sin ax.$

187. $\int x^n \cos ax\, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax\, dx.$

188. $\int \frac{x\, dx}{\sin^2 x} = - x \cot x + \log \sin x.$

189. $\int \frac{x\, dx}{\cos^2 x} = x \tan x + \log \cos x.$

190. $\int x e^{ax}\, dx = \frac{e^{ax}(ax - 1)}{a^2}.$

191. $\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx.$

192. $\int \frac{dx}{a + be^{rx}} = \frac{1}{ar} [rx - \log (a + be^{rx})].$

193. $\int \frac{dx}{ae^{rx} + be^{-rx}} = \frac{1}{r\sqrt{ab}} \tan^{-1}\left(e^{rx}\sqrt{\frac{a}{b}}\right).$

194. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}.$

195. $\int e^{ax} \cos bx dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2}.$

196. $\int x^n \log x\, dx = x^{n+1}\left[\frac{\log x}{n + 1} - \frac{1}{(n + 1)^2}\right].$

197. $\int (\log x)^n\, dx = x(\log x)^n - n \int (\log x)^{n-1}\, dx.$

198. $\int \frac{(\log x)^r\, dx}{x} = \frac{(\log x)^{r+1}}{r + 1}$, if $r \neq -1$.

199. $\int \frac{dx}{x \log x} = \log (\log x).$

200. $\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

201. $\int \cos^{-1} x\, dx = x \cos^{-1} x - \sqrt{1 - x^2}.$

202. $\int \tan^{-1} dx = x \tan^{-1} x - \log \sqrt{1 + x^2}.$

203. $\int \cot^{-1} x \, dx = x \cot^{-1} x + \log \sqrt{1 + x^2}.$

204. $\int \sec^{-1} x \, dx = x \sec^{-1} x - \log (x + \sqrt{x^2 - 1})$
$= x \sec^{-1} x - \cosh^{-1} x.$

205. $\int \csc^{-1} x \, dx = x \csc^{-1} x + \log (x + \sqrt{x^2 - 1})$
$= x \csc^{-1} x + \cosh^{-1} x.$

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